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DISCUSSIONS
APPLICATIONS FOR ADMISSION AND TRANSFER

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Relation Between Rail and Waterway Transportation: A Symposium	Mar., 1930	
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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

AN ANALYSIS OF MULTIPLE-SKEW ARCHES ON ELASTIC PIERS

By J. Charles Rathbun,[1] M. Am. Soc. C. E.

Synopsis

The purpose of this paper is to extend the analysis of the ring of a skew arch that was given in a previous paper[2] to include skew arches of more than one span and, by placing the skew equal to zero, to show how this method may be applied to the multiple right arch on elastic piers. In this latter case, the effect of eccentric loads is brought out by the formulas.

With the idea of being consistent with the method of analysis used in the first paper, the simple method of deflections was retained instead of using one of the more elegant methods, such as virtual work, or a procedure of a similar nature. In order that this paper may be more complete in itself an outline of the method of analysis of the single-span skew arch will be reviewed briefly. For the derivation of the formula for the deflection of a skewed cantilever under the various loads used, reference should be made to this previous paper.

As the use of the integral signs has had a tendency to confuse some of the designing engineers who might otherwise use this theory, the deflection formula is also given as a summation of a series of elastic blocks, the designer dividing the ring in somewhat the same manner as is customary in designing right arches. In this latter case, the terms due to ring-shortening, deformations due to shear, both radial and horizontal, and the terms containing the moment of inertia about the normal to the neutral surface may be omitted. Except possibly in rings of very unusual shapes, these terms are small, and the effect of omitting them will not be as great as the approximation of dividing the ring into blocks of finite lengths. In the case of ribbed arches the latter term, however, may reach an appreciable value.

Note.—Written discussion on this paper will be closed in August, 1931, *Proceedings*.

[1] Associate Prof. of Civ. Eng., Coll. of the City of New York, New York, N. Y.

[2] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 611.

The paper is divided into four parts:

Part I.—Outline of the Theory of the Skew Arch and a description of the method of extending it to two or more spans. By placing the angle of skew equal to zero the effect of eccentrically placed loads on a multiple right arch is found, and the results are summarized briefly.

Part II.—Method of Computing the Unit Stresses that occur in the concrete, and the reinforcing steel required.

Part III.—Application of the Theory to the Analysis of a Skew Arch with approach spans for concentrated loads.

Part IV.—An Experiment to Verify the Computations of Part III.

The notation used in the paper is given in Appendix I; the formulas for the skew cantilever are given in Appendix II, and those for the two-hinged single-span skew arch ring, in Appendix III.

It is believed that the examples of Part III should make clear the method of dividing the multiple-skew arch on elastic piers into a series of skew cantilevers that are determinate structures and can be designed by the ordinary methods. The unusual features of these computations are given in Part III, while the method of reducing them to the forces, and of reducing moments to unit stresses, is given in Part II.

Introduction

The multiple-span skew arch bridge will be found to be a solution for the case shown in Figs. 1 and 2, in which the roadways are parallel on both sides of the crossing. By using three skew spans with the piers between consisting of posts, the curves at the crossing can be made comparatively easy and the line of sight correspondingly long.

In adopting this type of structure the wing-walls that are usually used to retain the fill are replaced by approach spans (see Fig. 2). These spans not only present a better appearance, but they also increase the line of sight in many cases. The extra cost of the approach span will approximately be offset by the saving in the cost of the wing-walls.

The ring of a skew arch is a statically indeterminate structure containing forces and moments in such positions that, in general, all cannot be made parallel to a single plane. It must be analyzed, therefore, in three dimensions instead of the two that are used in the right arch. However, for the preliminary analysis for dead load, or any other loading in which the funicular polygon may be made to follow the arch ring approximately (as viewed in the direction parallel to the barrel of the arch), and in which the loads are placed symmetrically with respect to a plane midway between spandrel walls, a section parallel to the spandrels can be used. The dead load and total uniform load usually approximate these requirements. This same situation prevails with respect to the multiple skew with elastic piers. The skew introduces certain forces and moments not present in right arches for loads on the center line of the upper roadway. As will be shown, they are present for loads that are outside this central plane. The width of the arch ring has to

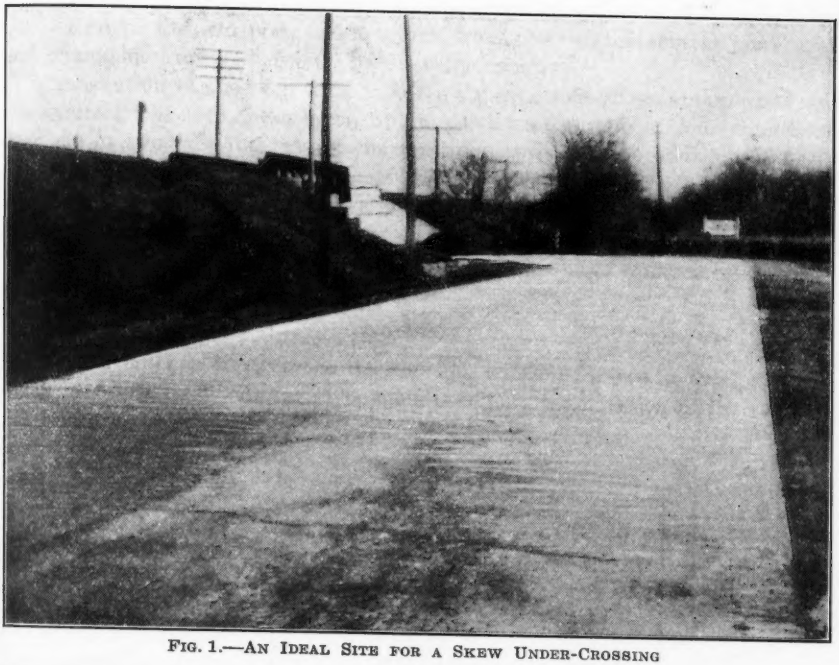


FIG. 1.—AN IDEAL SITE FOR A SKEW UNDER-CROSSING

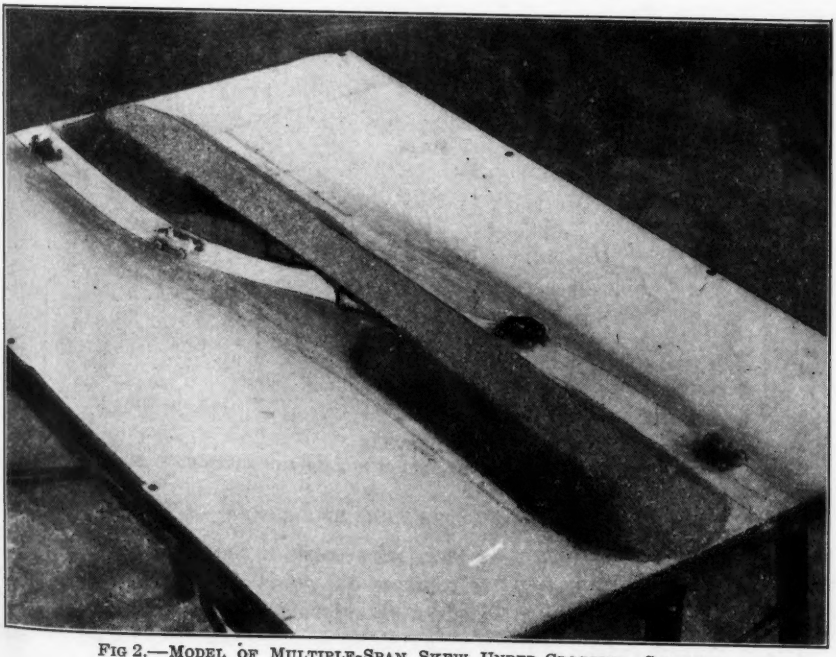


FIG 2.—MODEL OF MULTIPLE-SPAN SKEW UNDER-CROSSING, SHOWING INCREASED LINE OF VISION

be taken into consideration as well as the lateral position of the loads. As the design of the end piers often differs from that of the central piers, the end or outside arch will usually be unsymmetrical.

The solution of the skew arch involves six equations in six unknowns. In the symmetrical arch these break up into two groups, one of four equations and one of two. In the case of the multiple-skew arch the addition of each pier will involve the addition of six more unknowns. If the corresponding deflection equations are written, and are solved simultaneously, the labor is very great even if the number is reduced by symmetry. Although it is lengthy, the method herein outlined avoids the solution of a large number of simultaneous equations.

Notation

For convenience of reference the notation adopted for use in this paper, is given in Appendix I. As far as possible it follows the notation of the writer's previous paper.

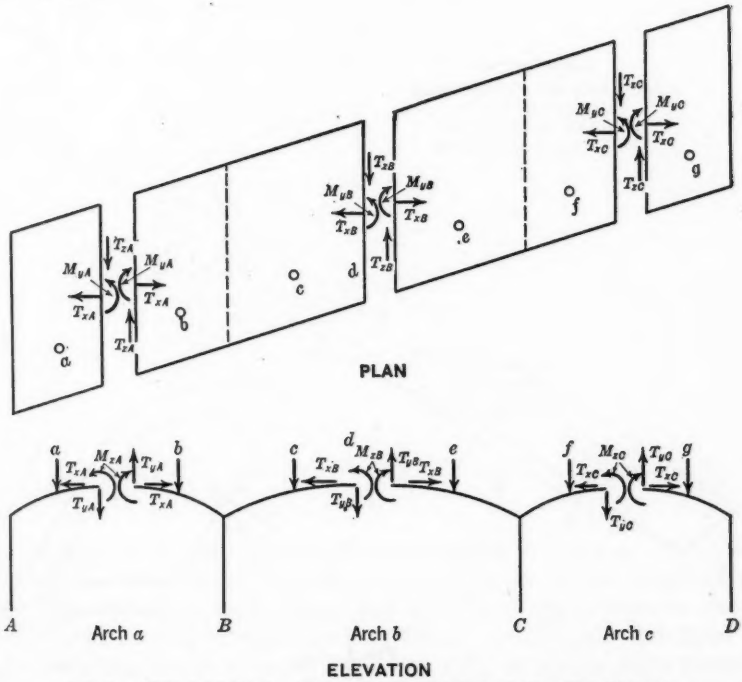


Fig. 3.—Multiple-Skew Arch Bridge Divided by Cutting Planes

PART I.—Outline of the Theory of the Skew Arch

Let Fig. 3 represent a three-span, skew-arch bridge on vertical elastic piers (*A*, *B*, *C*, and *D*) which are assumed to be rigidly supported at their bases. The arch can be considered as loaded with concentrated loads. The ring can be divided into sections, as shown, by cutting planes, making a series

of skewed cantilevers. If the arch ring has symmetry about any plane, as is usually the case, some labor can be saved by choosing that plane as the cutting plane.

First, consider the cantilever at the left, the one containing Pier *A*. This part is acted upon by the external loads and by three unknown forces and three unknown moments at the cutting plane. After the values of these six unknown quantities are obtained the stresses at any plane section of the cantilever can be determined by the usual processes of static mechanics from a diagram similar to Fig. 4.

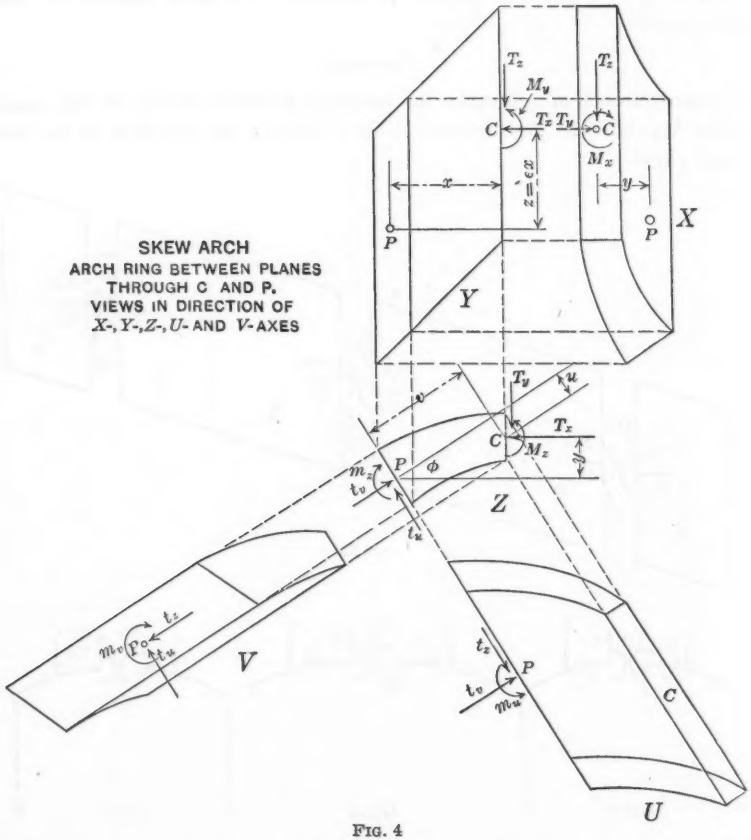


FIG. 4

The deflection at the cutting plane of this cantilever can be computed from Equations (20) (see Appendix II), the unknown reactions being carried as algebraic quantities, which are to be solved subsequently. The derivation of this formula is not given because it was developed in full in the paper on the single-span skew arch.[3] The origin of co-ordinates has been taken at the center of the cut section and the usual rectangular co-ordinate system used. The

[3] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 640.

method of choosing the co-ordinate system and the meaning of a part of the notation are given in Fig. 5. In order to shorten the formulas the terms, v and u, are introduced to replace the expressions:

$$v = x \cos\phi + y \sin\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and,

$$u = x \sin\phi - y \cos\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If the arch is of a mathematical shape, Equations (20) may be used as written if the integrations present no difficulties. However, in most designs, the arch ring is of a shape that does not lend itself to this form of treatment, and the analyst must divide the ring into a finite number of blocks by means of

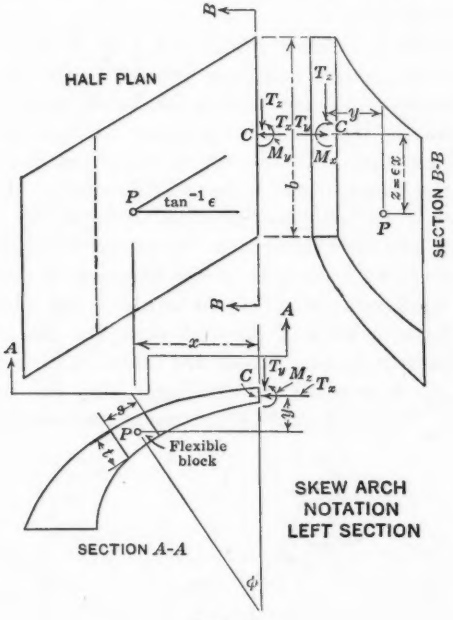


FIG. 5

planes parallel to the arch barrel and perpendicular to the ring. This process is exactly the same as that used in common practice in the design of right arches. The lengths of the blocks, s, may be taken as any convenient quantity, but they need not be the same for each block. In this case Equations (20) can be rewritten in the form of Equations (21), in which, finite blocks are used; the integral is replaced by the summation sign. Those terms that are normally small may be neglected, giving Equations (22). It is suggested that the designer check the writer's judgment in the matter of neglecting terms for any arch that he may be called upon to design. These neglected terms represent deformations due to shear, rib-shortening, and rotation about the radial axis. In rib arches the rotation, particularly, may be quite large. For the various types of loading terms it is suggested that the previous paper be consulted. The terms for vertical, dead load, uniform load, temperature,

and concentrated load at a point, x' z', are given. The terms due to grade and horizontal earth pressure are not repeated.

If Equations (21) are used the length, thickness, co-ordinate of the center, as well as the slope of the arch ring, can be obtained by scaling. The values of u and v can also be obtained from the layout, or they may be computed.

The deflection of this cantilever is expressed in terms of its components along each of the three axes and its rotation about each of these axes. Thus, six quantities, or six equations of deflection, are set up, which involve the six unknown quantities at the cutting plane; namely, the three components of the force and the three components of the moment. These quantities of force and moment are independent of each other. This is also true of the six quantities of deflection.

Next, consider the cantilever containing Pier B in Fig. 3. Except for the effect of the arch, b, this cantilever can be treated in exactly the same way, and the deflection of the cut section can be obtained. The forces at this section are the same as those of the cantilever, A, but opposite in direction, being action and reaction. As the abutments of each cantilever are fixed, the movement of the crown of one is necessarily equal to the movement of the crown of the other in each of the directions or rotations, because they must coincide both before and after deflecting. These six conditions can be expressed as six equations, care being taken as to the selection of the sign of the movement. Thus, six equations are set up in terms of the known external loads, certain physical characteristics of the arch ring, and the six unknown literal reactions at the cutting plane. These are taken so that they are mutually independent. As all these equations are linear, they can be solved by simple algebraic methods. The mathematical treatment may be lengthy, but it is not involved.

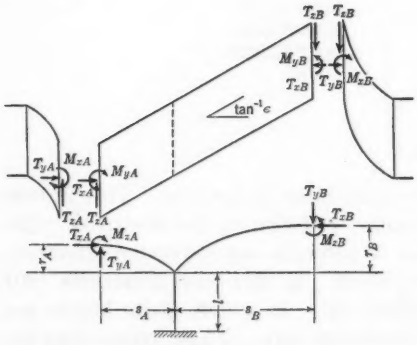


FIG. 6.—DIRECTIONS OF POSITIVE FORCES AND MOMENTS

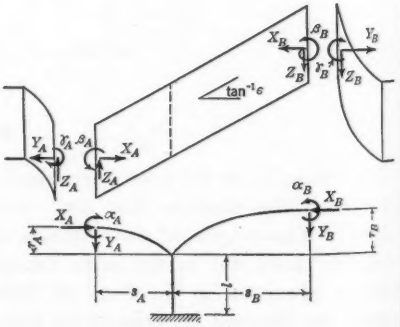


FIG. 7.—DIRECTIONS OF POSITIVE MOMENTS

If the skew structure consists of one span only, the indeterminate feature of the problem is complete. If another arch, such as Arch b in Fig. 3, is present, in computing the deflection of the right cantilever of Arch a, recognition must be taken of the deflection caused by the loads from this second

arch. These loads consist of the six unknown components of the reaction at the cutting plane and the external loads between the cutting plane of Arch *b*, and the pier, *B*. By examining the forces acting on this cantilever in Fig. 6 and the direction of the movement of the crown in Fig. 7, one can write the following equations of deflection:

For the right side:

$$R_x = -\left[\int_0^l \frac{\lambda}{A}\,dy + \int_0^l (y + r_A)(y + r_B)\frac{dy}{I_z} - S_A\,S_B\,\varepsilon^2\,\lambda\int_0^l \frac{dy}{F}\right] T_{xB} + \left[S_B\int_0^l (y + r_A)\frac{dy}{I_z}\right] T_{yB} - \left[S_A\,S_B\,\varepsilon\,\lambda\int_0^l \frac{dy}{F}\right] T_{zB} - \left[\int_0^l (y + r_A)\frac{dy}{I_z}\right] M_{zB} + \left[S_A\,\varepsilon\,\lambda\int_0^l \frac{dy}{F}\right] M_{yB} + \left[S_B\int_0^l (y + r_A)\frac{dy}{I_z}\right]\sum L - \left[\int_0^l (y + r_A)\frac{dy}{I_z}\right]\sum L\,x' \ldots (3a)$$

$$R_y = \left[S_A\int_0^l (y + r_B)\frac{dy}{I_z}\right] T_{xB} + \left[\int_0^l \frac{dy}{A} - S_A S_B\int_0^l \frac{dy}{I_z} - S_A\,S_B\,\varepsilon^2\int_0^l \frac{dy}{I_y}\right] T_{yB} + \left[S_A\,\varepsilon\int_0^l (y + S_B)\frac{dy}{I_y}\right] T_{zB} + \left[S_A\int_0^l \frac{dy}{I_z}\right] M_{zB} - \left[S_A\,\varepsilon\int_0^l \frac{dy}{I_y}\right] M_{xB} + \left[\int_0^l \frac{dy}{A} - S_A\,S_B\int_0^l \frac{dy}{I_z} - S_A\,S_B\,\varepsilon^2\int_0^l \frac{dy}{I_y}\right]\sum L + \left[S_A\int_0^l \frac{dy}{I_y}\right]\sum L\,z' + \left[S_A\int_0^l \frac{dy}{I_z}\right]\sum L\,x' \ldots\ldots\ldots\ldots\ldots (3b)$$

$$R_z = -\left[S_A\,S_B\,\varepsilon\,\lambda\int_0^l \frac{dy}{F}\right] T_{xB} + \left[S_B\,\varepsilon\int_0^l (y + r_A)\frac{dy}{I_y}\right] T_{yB} - \left[\lambda\int_0^l \frac{dy}{A} + \int_0^l (y + r_B)(y + r_A)\frac{dy}{I_y} - S_A\,S_B\,\lambda\int_0^l \frac{dy}{F}\right] T_{zB} - \left[S_A\,\lambda\int_0^l \frac{dy}{F}\right] M_{yB} + \left[\int_0^l (y + r_A)\frac{dy}{I_y}\right] M_{xB} + \left[S_B\,\varepsilon\int_0^l (y + r_A)\frac{dy}{I_y}\right]\sum L - \left[\int_0^l (y + r_A)\frac{dy}{I_y}\right]\sum L\,z' \ldots (3c)$$

$$R_\alpha = -\left[\int_0^l (y + r_B)\frac{dy}{I_z}\right] T_{xB} + S_B\left[\int_0^l \frac{dy}{I_z}\right] T_{yB} - \left[\int_0^l \frac{dy}{I_z}\right] M_{zB} + S_B\left[\int_0^l \frac{dy}{I_z}\right]\sum L - \left[\int_0^l \frac{dy}{I_z}\right]\sum L\,x' \ldots (3d)$$

$$R_\beta = \left[S_B\,\varepsilon\,\lambda\int_0^l \frac{dy}{F}\right] T_{xB} - \left[S_B\,\lambda\int_0^l \frac{dy}{F}\right] T_{zB} + \left[\lambda\int_0^l \frac{dy}{F}\right] M_{yB} \ldots (3e)$$

and,

$$R_\gamma = \left[- S_B \varepsilon \int_0^l \frac{dy}{I_y} \right] T_{y\,B} + \left[\int_0^l (y + r_B) \frac{dy}{I_y} \right] T_z - \left[\int_0^l \frac{dy}{I_y} \right] M_x - \left[S_B \varepsilon \int_0^l \frac{dy}{I_y} \right] \sum L + \left[\int_0^l \frac{dy}{I_y} \right] \sum L z' \ldots\ldots\ldots\ldots (3f)$$

For the left side :

$$L_x = - \left[\int_0^l \frac{\lambda}{A}\, dy + \int_0^l (y + r_A)(y + r_B) \frac{dy}{I_z} - S_A S_B \varepsilon^2 \lambda \int_0^l \frac{dy}{F} \right] T_{x\,A} - \left[S_A \int_0^l (y + r_B) \frac{dy}{I_z} \right] T_{y\,A} - \left[S_A S_B \varepsilon \lambda \int_0^l \frac{dy}{F} \right] T_{z\,A} - \left[\int_0^l (y + r_B) \frac{dy}{I_z} \right] M_{z\,A} - \left[S_B \varepsilon \lambda \int_0^l \frac{dy}{F} \right] M_{y\,A} + \left[S_A \int_0^l (y + r_B) \frac{dy}{I_z} \right] \sum L - \left[\int_0^l (y + r_B) \frac{dy}{I_z} \right] \sum L x' \ldots (4a)$$

$$L_y = \left[S_B \int_0^l (y + r_A) \frac{dy}{I_z} \right] T_{x\,A} + \left[S_A S_B \int_0^l \frac{dy}{I_z} - \int_0^l \frac{dy}{A} + S_A S_B \varepsilon^2 \int_0^l \frac{dy}{I_y} \right] T_{y\,A} + \left[S_B \varepsilon \int_0^l (y + r_A) \frac{dy}{I_y} \right] T_{z\,A} + \left[S_B \int_0^l \frac{dy}{I_z} \right] M_{z\,A} - \left[S_B \varepsilon \int_0^l \frac{dy}{I_y} \right] M_{x\,A} - \left[S_A S_B \int_0^l \frac{dy}{I_z} - \int_0^l \frac{dy}{A} + S_A S_B \int_0^l \frac{dy}{I_y} \right] \sum L + \left[S_B \int_0^l \frac{dy}{I_z} \right] \sum L x' + \left[S_B \varepsilon \int_0^l \frac{dy}{I_y} \right] \sum L z' \ldots\ldots\ldots (4b)$$

$$L_z = - \left[S_A S_B \varepsilon \lambda \int_0^l \frac{dy}{F} \right] T_{x\,A} - \left[S_A \varepsilon \int_0^l (y + r_B) \frac{dy}{I_y} \right] T_{y\,A} + \left[S_A S_B \lambda \int_0^l \frac{dy}{F} - \lambda \int_0^l \frac{dy}{A} - \int_0^l (y + r_A)(y + r_B) \frac{dy}{I_y} \right] T_{z\,A} + \left[S_B \lambda \int_0^l \frac{dy}{F} \right] M_{y\,A} + \left[\int_0^l (y + r_B) \frac{dy}{I_y} \right] M_{x\,A} + \left[S_A \varepsilon \int_0^l (y + r_B) \frac{dy}{I_y} \right] \sum L - \left[\int_0^l (y + r_B) \frac{dy}{I_y} \right] \sum L z' . (4c)$$

$$L_\alpha = - \left[\int_0^l (y + r_A) \frac{dy}{I_z} \right] T_{x\,A} - \left[S_A \int_0^l \frac{dy}{I_z} \right] T_{y\,A} - \left[\int_0^l \frac{dy}{I_z} \right] M_{z\,A} + \left[S_A \int_0^l \frac{dy}{I_z} \right] \sum L - \left[\int_0^l \frac{dy}{I_z} \right] \sum L x' \ldots\ldots\ldots (4d)$$

$$L_\beta = \left[S_A \varepsilon \lambda \int_0^l \frac{dy}{F} \right] T_{x\,A} - \left[S_A \lambda \int_0^l \frac{dy}{F} \right] T_{z\,A} - \left[\lambda \int_0^l \frac{dy}{F} \right] M_{y\,A} \ldots (4e)$$

and,

$$L_\gamma = \left[S_A \varepsilon \int_0^l \frac{dy}{I_y}\right] T_{y\,A} + \left[\int_0^l (y + r_A)\frac{dy}{I_y}\right] T_{z\,A} - \left[\int_0^l \frac{dy}{I_y}\right] M_{x\,A} - \left[S_A \varepsilon \int_0^l \frac{dy}{I_y}\right] \sum L - \left[\int_0^l \frac{dy}{I_y}\right] \sum L\, z' \ldots\ldots\ldots (4f)$$

It is to be noted in this cantilever that, except for forces and moments at the cut sections, the section from the crown of Arch a to the top of the pier has no force from Arch b acting on it, and also that the deformation of the section from the crown of Arch b to the top of the pier does not influence the deflection of the crown of Arch a. Therefore, the summation of the terms of deflection need be taken over the pier only. As this pier is usually vertical and straight, the formulas are written with this condition in mind. The deflections thus obtained must be added to those already computed for the right half of Arch a, before combining it with those of the left half to obtain the six equations. Incidentally, the deflection of the cut section of Arch b, due to the forces at the cut section of Arch a, can be written at once from Equations (3) producing Equations (4).

Solving the six equations obtained for Arch a, the values of the reactions at the crown of this arch are obtained in terms of all loads between the section of Arch b and the abutment of Pier A, and of the six reactions at the section of Arch b.

The cantilever consisting of Pier B and the left portion of Arch b can now be taken up in the same way as the cantilever of Arch a. The deflection of this cantilever is caused by the loads acting directly upon it and the reactions from the section of Arch a, as well as the six unknowns from the section of Arch b. For the deflection due to the loads on the ring of Arch b, Equations (20) or (21) in Appendix II should be used, while for the deflection due to the reactions from Arch a, Equations (4) are to be taken. This deflection can thus be written in terms of the external loads and their reactions from the cut sections of Arch a and Arch b. The reactions from Arch a involve the unknown reactions from Arch b. Writing the deflections, equations similar to those for the left cantilever of Arch a are obtained after collecting the terms. No operations of a higher degree than the first are involved (six first-degree equations in six unknowns), in obtaining the reactions at the crown of Arch b. These represent the deflections of the system consisting of all of Arch a and the portion of Arch b to the left of the section.

If the system is symmetrical about the crown of the center span, as is true in the example given, the deflections of the arch on the right and the right side of Arch b will involve the same equations, except possibly for loading terms. Equating these six deflections against their corresponding six on the left half of the system gives four equations in four unknowns and two equations in two unknowns, which when solved fully determine the reactions in the ring of Arch b. If, however, the ring is not symmetrical, Arch a, together with the left portion of Arch b, can be treated in the same manner that was used on the left part of Arch a. The right part of Arch b is treated in the same way that was used on the right part of Arch a. In this case the reactions

of the section of Arch *b* will be found in terms of the loads and the reactions from Arch *c*. This process must be continued until the arch on the extreme right is reached, when the right cantilever will have no unknown loads acting on it.

Having obtained the reaction of the right-hand arch, or the center arch in the case of symmetry, the reaction of the arch to the left can be found by substituting these values in the reaction at the cut section of this second arch. The process is continued until the left arch, or Arch *a*, is reached.

The entire system is now broken up into a system of cantilevers acted upon by known loads. By means of Fig. 4 and Equations (5), which are presented under "Method of Computing Unit Stresses", the stresses on any other section can be obtained.

It is interesting to note that if the skew is made equal to zero (a right multiple arch), the crown forces and moments, T_x, T_y, and M_z, are independent of the three remaining forces and moments. The six equations break up into two groups of three equations each. This independence remains throughout the entire set of computations. (Note Equations (30), (31), (32), and (33) which develop from Equations (20), (24), (3), and (4), by placing ϵ equal to zero.) As a result the equations and method herein given can be used to study the effect of loads that are not in the center of the upper roadway on a multiple arch with elastic piers, or, in fact, the effect on a single span right arch.

This procedure also gives a method of analysis for a right multiple arch on elastic piers which at no time involves the solution of more than three equations in three unknowns in one group. A discussion on the relative merits of this solution of the multiple arch and the others known to the profession would be valuable.

PART II.—Method of Computing the Unit Stresses

As the writer showed[4] in the previous paper, the polygon of forces follows the central line of the skew arch as closely as possible. In an arch of the ideal shape, therefore, it follows this line exactly, and the distribution of forces and moments along the cutting plane, such as shown in Fig. 8, or the crown section, is uniform; that is, the thrust is uniformly distributed along this section while the shear is non-existent. If the adjustment is not so perfect the thrust can still be taken as uniformly distributed. Consider, therefore, any section at a point, *P*, Fig. 4. The computed components of the force on this section are t_v, t_u, and t_z, and the moments are m_z, m_u, and m_v. The force can also be resolved into three components: One parallel to the spandrel wall and perpendicular to the radius, t; another parallel to the radius, t_u; and the third, perpendicular to these two forces, s. The moment can be resolved into components around these axes. The main reinforcement steel is parallel to the force, t. This force can be considered as uniformly distributed over a section the area of which is the projected area of the section in the direction of this force, t, proper allowance being made for steel area. The shear, s,

[4] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 611.

can be distributed over this same projected section using the parabolic distribution. This gives zero shear at the edges. At this place the force, of necessity, must be parallel to the spandrel walls. Three halves of the average shear value is the maximum at the center. These values will usually not be great enough to require reinforcement.

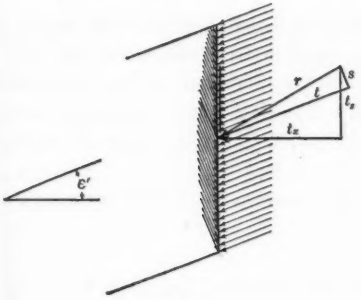


FIG. 8

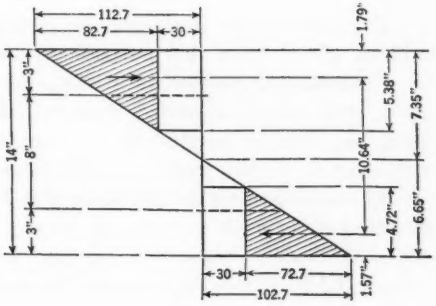


FIG. 9

The moment, m, about the axis parallel to s is treated in the same way as the force, t; it acts directly on the steel, and the projected area of the concrete is used in computing width. The torque, f, about the axis parallel to t will usually be small in value, but it can be computed on the projected section. The distribution of this stress is parabolic along each side of the section, with a maximum value in the center of the long side, and a zero value at the corners.

As the main reinforcing steel is parallel to the spandrel walls, the thrust and moment acting on this steel are next computed, as are the shear and thrust at the maximum point of the section; that is, the center of the top and bottom sides.

In making these computations after having obtained the crown reactions, the following formulas are used. Equations (5) are set up directly from Fig. 4, by taking forces and moments in the direction of and about the various axes.

$$t_v = T_x \cos\phi + T_y \sin\phi + V \sin\phi - H \cos\phi \quad (5a)$$

$$t_u = -T_x \sin\phi + T_y \cos\phi + V \cos\phi + H \sin\phi \quad (5b)$$

$$t_z = T_z \quad (5c)$$

$$m_z = T_x y - T_y x + M_z - M_{Lz} \quad (5d)$$

$$m_u = T_x z \cos\phi + T_y z \sin\phi - T_z v + M_y \cos\phi + M_x \sin\phi + \varepsilon M_{Lz} \sin\phi \quad (5e)$$

and,

$$m_v = -T_x z \sin\phi + T_y z \cos\phi + T_z u - M_y \sin\phi + M_x \cos\phi + \varepsilon M_{Lz} \cos\phi \quad (5f)$$

For brevity substitute ϵ' for $\tan^{-1}\epsilon'$; then, by resolving forces about a set of axes making an angle of ϕ with the original set of u, v, z:

$$\varepsilon' = \varepsilon \cos\phi \quad (6)$$

$$t = t_v \cos \varepsilon' + t_z \sin \varepsilon' \quad \ldots\ldots (7a)$$

$$s = t_v \sin \varepsilon' - t_z \cos \varepsilon' \quad \ldots\ldots (7b)$$

$$m = m_z \cos \varepsilon' - m_v \sin \varepsilon' \quad \ldots\ldots (7c)$$

and,

$$f = m_z \sin \varepsilon' + m_v \cos \varepsilon' \quad \ldots\ldots (7d)$$

Example 1.—To illustrate the method here outlined a numerical example will be given. For this purpose an arch is chosen other than the one used to illustrate the method of computing the crown reactions in the multiple-skew arch, which is tested with the model. The following quantities have been measured or computed: $\epsilon = \tan 47° 11' = 1.079$; $b = 45.6$ ft.; crown thickness = 14 in.; and $b \cos 47° 11' = 31$ ft. = 372 in.

By methods described in Appendix II of this paper, the crown stresses are as listed in Table 1.

TABLE 1.—THRUST AND MOMENTS AT THE CROWN; EXAMPLE 1

Load	T_x	T_z	M_z	M_x
Dead	259.33	273.43	262.94	−234.37
Temperature	−37.92	−55.71	145.93	− 10.28
Uniform	6.16	6.04	20.50	− 16.21
Concentrated	13.80	13.44	104.20	−101.08

At the crown, $\phi = 0$; $\cos \phi = 1$; $\sin \phi = 0$; $\varepsilon = \varepsilon' = 1.079$; $\cos \varepsilon' = 0.6796$; and $\sin \epsilon' = 0.7335$. Substituting these values in Equation 7,

$$t = 259.33 \times 0.6796 + 273.43 \times 0.7335 = 176.35 + 200.57 = 376.92$$

$$s = 259.33 \times 0.7335 - 273.43 \times 0.6796 = 190.22 - 185.33 = 4.89$$

$$m = 262.94 \times 0.6796 + 234.37 \times 0.7335 = 178.71 + 171.92 = 350.63$$

$$f = 262.94 \times 0.7335 - 234.37 \times 0.6796 = 192.87 - 159.29 = 35.58$$

In a similar manner the stresses for the other loads may be computed. The values are as shown in Table 2.

TABLE 2.—STRESSES DETERMINED BY EQUATIONS (7); EXAMPLE 1

Load	t	s	m	f
Dead	376.92	4.89	350.63	33.58
Temperature, 50°	68.64	− 10.04	106.73	100.06
Uniform	8.62	0.42	25.80	4.00
Concentrated	19.28	1.04	140.04	7.76
Total with 50° drop	336.18	16.39	628.19	145.40
Total with 30° rise	446.00	0.32	457.47	− 14.66

It is now necessary to design a section, 14 in. thick and 372 in. wide, to carry the thrusts and moments in Table 2. The design for t and m presents no unusual problem, being the simple case of an eccentric load on a column.

The horizontal shear at the center equals $\frac{16.39 \times 1.5}{14 \times 372} = 5.0$ lb. per sq. in. Shear due to torque, or f, using the formula for torque,

$$v = \frac{9\,f}{2\,b\,d^2} \qquad (8)$$

is equal to $\frac{9 \times 145.40 \times 12}{2 \times 372 \times 14^2} = 107.7$ lb. per sq. in., and the shearing stress on the section equals $\pm\ 107.7 + 5.0 = -102.7$, or $+\ 112.7$ lb. per sq. in.

Thus, at the upper surface of the crown there is a shear stress of 112.7 lb. per sq. in., while on the lower surface the shear is 102.7 lb. per sq. in. Assuming a straight-line variation of shear between surfaces, Fig. 9 represents this distribution. If 30 lb. per sq. in. is allowed for concrete, a force represented by the volume of a wedge (base, 82.7 lb. per sq. in., height, 5.38 in., and depth, 12 in.) must be taken up by reinforcement on the upper surface. On the lower surface the force is likewise represented by the volume of a wedge, $\frac{72.7 \times 4.72 \times 12}{2}$. This reinforcement must be adequate to take the torque of these two forces when acting through a lever arm equal to the distance apart of the two rows of steel, in this case, 8 in.

Noting that the center of gravity of the triangle is one-third the distance from the base to the altitude and using a steel stress of 22 000 lb. per sq. in.,

$$\frac{82.7 \times 5.38 \times 12 \times 10.64}{2 \times 8 \times 22\,000} = 0.1614 \text{ sq. in.}$$

and,

$$\frac{72.7 \times 4.72 \times 12 \times 10.64}{2 \times 8 \times 22\,000} = 0.1245 \text{ sq. in.}$$

This steel is laid at right angles to the main reinforcement and if considered advisable, may be reduced toward the spandrel walls, since the shear is distributed according to the ordinates of a parabola.

The stresses in any other section than the crown present no unusual difficulties that have not been clarified in the foregoing computations.

PART III.—Application of the Theory to the Analysis of a Skew Arch

In order to illustrate the application of this theory to the analysis of a structure, Fig. 10 will be taken as an example. The entire set of computations will not be given, but it is hoped that the outline will prove sufficiently complete to be followed readily by those familiar with the method as applied to the single-span skew arch.

Example 2.—The arch rings, including the outside piers (Fig. 10), have been divided into blocks of finite length and, by scaling, the values listed in Table 3 have been obtained. Furthermore, from Fig. 10, $b = 42$ ft.; $\varepsilon = \tan 34° 55' = 0.698$; and $\lambda = 2.5$.

In computing temperature the loading terms for Arch a are taken as:

$$L_x + R_x = 42 \times 23.5 \times 0.0000065 \times 2\,000 \times 144 = 1\,848$$

and,

$$L_z + R_z = \varepsilon \times 1\,848 = 1\,290$$

and for Arch *b* these terms are:

$$L_x = 42 \times 26.5 \times 0.0000065 \times 2\,000 \times 144 = 2083.5$$

and,

$$L_z = \varepsilon \times 2083.5 = 1454.31$$

TABLE 3.—SCALES, VALUES, AND COMPUTATIONS ACCOMPANYING EXAMPLE 7

Point	x	y	u	t	v	s	ϕ	x	y	t	u	s	v	ϕ
	LEFT HALF OF ARCH *a*							LEFT HALF OF ARCH *b*						
1	0.7	0.00	0.00	1.02	0.70	1.43	0.031	1.30	0.00	1.50	0.00	2.58	1.30	0.000
2	2.14	0.07	0.06	1.14	2.14	1.43	0.069	3.86	0.07	1.54	0.12	2.58	3.90	0.047
3	3.55	0.19	0.16	1.33	3.55	1.43	0.099	6.46	0.22	1.63	0.22	2.58	6.47	0.067
4	4.97	0.33	0.25	1.63	5.00	1.43	0.119	9.02	0.43	1.76	0.43	2.58	9.04	0.095
5	6.42	0.52	0.45	2.05	6.45	1.43	0.151	11.59	0.71	1.96	0.72	2.58	11.62	0.124
6	7.82	0.77	0.73	2.55	7.85	1.43	0.191	14.16	1.10	2.16	1.15	2.58	14.16	0.160
7	9.25	1.07	1.13	3.17	9.26	1.43	0.238	16.70	1.50	2.42	1.45	2.58	16.73	0.177
8	10.70	1.82	5.33	3.00	9.45	1.84	0.810	19.25	2.00	2.74	1.93	2.58	19.26	0.206
9	11.96	2.80	9.34	2.80	7.95	1.35	1.915	21.75	2.55	3.08	2.53	2.58	21.75	0.238
10	11.88	4.60	12.05	2.80	4.11	2.20	−22.000	24.20	3.06	3.67	1.30	2.50	24.36	0.186
11	11.82	6.80	12.05	2.62	6.32	2.20	−22.000	25.95	3.40	4.50	0.42	1.05	26.20	0.151
12	11.72	9.00	12.05	2.43	8.53	2.20	−22.000	(26.50)	(3.50)	....	....	....	.....	.....
13	11.63	11.20	12.05	2.24	10.75	2.20	−22.000	.....	.....	....	....	....	.....	.....
14	11.55	13.40	12.05	2.05	13.96	2.20	−22.000	.....	.....	....	....	....	.....	.....
	RIGHT HALF OF ARCH *a* (POINTS 1 TO 7 ARE THE SAME AS THE LEFT SIDE)													
8	10.95	1.50	1.35	4.10	10.95	2.10	0.265	.....	.....	....	....	....	.....	.....

In both cases this is $b\,E$, or $42\,E$ times the expansion in the direction of X or Z. The multiple, 42, is used because each term has the common factor, 42, cancelled from the denominator (as is also E), or $2\,000 \times 144$ kips per sq. ft. The coefficient of expansion was taken as 0.0000065.

By computation from the scaled quantities in Table 3 the summations or approximations of the integrals used in the formulas are found to be as shown in Table 4. The summations are taken from the crown to the abutment, or from the load to the abutment in the case of partial summations for the left side of Arch *a*. In the case of the right side of Arch *a* and the left side of Arch *b*, the posts of the intermediate pier presented a problem somewhat different from the blocks. This section of the arch was omitted, therefore, and the effect of the posts was treated separately.

In this example the writer carried the computations to a larger number of significant figures than a practical designer would have done. This was done in order to study the results better. Furthermore, it required little more effort, because a multiplying machine was used. The values are published as computed, because it is felt that an experienced designer can balance time against accuracy as he sees fit, and would not take these computations as an example in this particular. Others may be interested in the numbers as computed.

The piers consist of five 3 by 3-ft. posts, 9 ft. 6 in. long and 9 ft. 8 in. on centers. They were assumed to be held rigidly, top and bottom. The

evaluations of the deflections are given in a few cases in the following computations. As the other computations for deflection of the post are of a similar nature, they are here omitted.

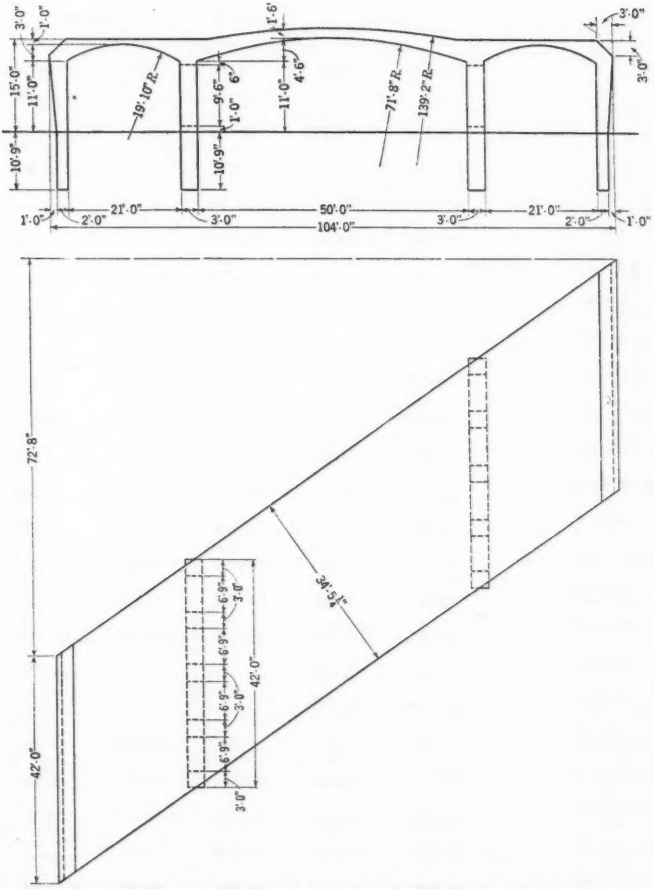


FIG. 10.—DIMENSIONAL DRAWING OF ARCH ANALYZED IN EXAMPLE 2

The deflection of five posts due to the moment about the Y-axis equals:
Shaft action:

$$T = \frac{E_s F}{l}\theta = \frac{180 \times 3^4 \times 5 \times \theta}{2.5 \times 9.5 \times 2.05 \times \pi \times (3^2 + 3^2)} = 2.383\,\theta$$

Cantilever action:

$$T = 2\,(19\tfrac{1}{3}^2 + 9\tfrac{2}{3}^2)\frac{3 \times 8 \times 3^4}{12 \times 9.5^3 \times 2}\theta = 88.281\,\theta$$

Total:

$$T = (2.383 + 88.281)\,\theta = 90.664\,\theta$$

TABLE 4.—Summation of Blocks in Example 2

Quantity	Summations, Abutment to Crown			Summations, Abutment to Load		
	Arch a		Arch b	Arch a		Arch b
	Left	Right	Left	Left	Right	Left
$\frac{s}{t}$	12.000	7.164	13.042	.....	.....	.....
$\frac{s}{t^3}$	4.504	3.615	3.633	1.254	0.365	0.751
$\frac{s\,v}{t}$	635.846	270.922	2 322.118	.....	.....	.....
$\frac{s\,u^2}{t}$	726.480	50.036	249.020	.....	.....	.....
$\frac{s \sin \phi}{t}$	6.050	1.118	1.729	.....	.....	.....
$\frac{s \sin^2 \phi}{t}$	5.292	0.456	0.548	5.262	0.426	0.504
$\frac{s \cos \phi}{t}$	6.757	6.764	12.606	.....	.....	.....
$\frac{s \cos^2 \phi}{t}$	6.703	6.703	12.500	2.128	2.128	4.783
$\frac{s \sin \phi \cos \phi}{t}$	0.932	0.772	1.377	0.595	0.435	0.894
$\frac{s\,x \sin \phi \cos \phi}{t}$	6.617	4.895	21.216	5.519	3.797	17.123
$\frac{s\,x^2 \sin \phi \cos \phi}{t}$	52.298	38.729	377.995	53.022	34.453	339.598
$\frac{s\,u \cos \phi}{t}$	3.750	2.339	9.806	3.287	1.876	7.689
$\frac{s\,u\,x \cos \phi}{t}$	22.300	18.883	166.927	20.454	17.038	147.347
$\frac{s\,x \cos^2 \phi}{t}$	29.524	29.506	137.303	17.813	17.795	90.280
$\frac{s\,x^2 \cos^2 \phi}{t}$	196.646	196.157	2 157.617	155.555	155.065	1 771.467
$\frac{s\,u \sin \phi}{t}$	61.566	4.467	.....	.....	.....	.....
$\frac{s\,u\,x \sin \phi}{t}$	718.482	52.134	.....	.....	.....	.....
$\frac{s\,x \sin^2 \phi}{t}$	61.348	4.957	12.565	61.234	4.846	12.156
$\frac{s\,x^2 \sin^2 \phi}{t}$	714.270	56.408	300.789	713.808	55.946	300.695
$\frac{s\,v \cos \phi}{t}$	29.005	30.030	139.483	.....	.....	.....
$\frac{s\,v\,x \cos \phi}{t}$	188.702	200.607	2 198.265	.....	.....	.....
$\frac{s\,v \sin \phi}{t}$	51.322	9.678	26.822	50.212	8.568	22.718
$\frac{s\,v\,x \sin \phi}{t}$	578.481	95.777	524.447	574.153	91.448	485.723
$\frac{s\,x \sin^2 \phi}{t^3}$	10.637	0.591	1.036	.....	.....	.....
$\frac{s\,x^2 \sin^2 \phi}{t^3}$	123.436	6.333	35.175	123.220	6.117	34.000
$\frac{s\,y}{t^3}$	9.075	1.056	2.433	8.783	0.764	1.806
$\frac{s\,y^2}{t^3}$	91.519	7.537	11.824	.....	.....	.....
$\frac{s\,x}{t^3}$	20.085	9.740	29.313	13.278	2.932	13.504
$\frac{s\,x^2}{t^3}$	166.272	45.830	374.946	145.375	24.933	253.737
$\frac{s\,x\,y}{t^3}$	102.496	9.330	44.138	101.399	8.234	38.445

TABLE 4.—(*Continued.*)

Quantity	Summations, Abutment to Crown			Summations, Abutment to Load		
	Arch *a*		Arch *b*	Arch *a*		Arch *b*
	Left	Right	Left	Left	Right	Left
$\frac{s\,u^2}{t^3}$	127.544	5.565	28.155	.....	.....	.....
$\frac{s \sin \phi}{t^3}$	1.182	0.303	0.325	.....	.....	.....
$\frac{s \sin^2 \phi}{t^3}$	0.927	0.064	0.074	.....	.....	.....
$\frac{s \cos \phi}{t^3}$	3.578	3.564	3.578	.....	.....	.....
$\frac{s \cos^2 \phi}{t^3}$	3.593	3.551	3.559	0.357	0.317	0.691
$\frac{s \sin \phi \cos \phi}{t^3}$	0.278	0.264	0.285	0.071	0.058	0.125
$\frac{s\,x \sin \phi \cos \phi}{t^3}$	1.166	1.033	3.523	0.590	0.458	2.252
$\frac{s\,x^2 \sin \phi \cos \phi}{t^3}$	7.143	5.782	52.870	5.110	3.749	41.459
$\frac{s\,u \cos \phi}{t^3}$	0.513	0.461	1.736	0.276	0.225	1.075
$\frac{s\,x \cos^2 \phi}{t^3}$	9.449	9.149	27.800	2.696	2.396	12.098
$\frac{s\,u\,x \cos \phi}{t^3}$	3.150	2.723	25.278	2.276	1.848	19.442
$\frac{s\,x^2 \cos^2 \phi}{t^3}$	42.845	39.502	339.906	22.160	18.817	219.740
$\frac{s\,u \sin \phi}{t^3}$	10.753	0.513	1.248	.....	.....	.....
$\frac{s\,u\,x \sin \phi}{t^3}$	125.199	5.812	30.379	.....	.....	.....

$$b\,\frac{\lambda\,s}{F} = \frac{42}{90.664} = 0.4632$$

$$\int y\,\frac{ds}{I_z} = \frac{12}{5 \times 3^4}\int_4^{13.5} y\,dy = \frac{12}{2 \times 5 \times 3^5}\,(13.5^2 - 4^2) = 2.463$$

$$\int y^2\,\frac{ds}{I_z} = \frac{12}{5 \times 3^4 \times 3}\,(13.5^3 - 4^3) = 23.668$$

The following values are obtained for the various terms for the pier of five posts, in which allowance is made for the elasticity of the base:

$$\frac{b\,\lambda\,s}{F} = 0.3330 + 0.4632 = 0.7962$$

$$\frac{b\,s}{I_y} = 0.00227 + 42 \times 0.001125 = 0.00227$$
$$+ 0.04725 = 0.04952$$

$$\frac{b\,s}{I_z} = 0.4444 + \frac{12 \times 42 \times 9.5}{3^4 \times 5} = 0.4444$$
$$+ 11.8222 = 12.2666$$

$$\int (y+4)\frac{b\,dy}{I_y} = 14 \times 0.00227 + \frac{b}{I_y}\int y\,dy + \frac{4\,b}{I_y}\int dy = 0.03175 + 0.41347 = 0.44522$$

and,

$$\int (y+5.75)\frac{b\,dy}{I_y} = 15.75 \times 0.00227 + 0.49616 = 0.53188$$

Terms containing I_z are similarly computed.

For simplicity of arrangement, the equations for deflection of the skewed cantilevers may be written in the following form:

$$A\,T_x + B\,T_y + C\,T_z + D\,M_z + E\,M_y + F\,M_x + L_x \quad \ldots\ldots (9a)$$
$$B\,T_x + G\,T_y + H\,T_z + I\,M_z + F\,M_y + K\,M_x + L_y \quad \ldots\ldots (9b)$$
$$C\,T_x + H\,T_y + L\,T_z + N\,M_y + O\,M_x + L_z \quad \ldots\ldots (9c)$$
$$D\,T_x + I\,T_y + P\,M_z + L_\alpha \quad \ldots\ldots (9d)$$
$$E\,T_x + F\,T_y + N\,T_z + Q\,M_y + R\,M_x + L_\beta \quad \ldots\ldots (9e)$$

and,

$$F\,T_x + K\,T_y + O\,T_z + R\,M_y + S\,M_x + L_\gamma \quad \ldots\ldots (9f)$$

The coefficients are evaluated from the summations as given in Equation (20) and are made up of the following terms:

$A = 6.703 + 13.230 + 1\,098.224 + 0.652 + 1.765 + 537.922$	=	+ 1 658.496
$B = -1.399 - 1\,229.946 + 0.189 - 0.141 - 31.277$	=	— 1 262.574
$C = -0.896 - 2.543 - 781.677$	=	— 785.118
$D =$	=	108.900
$E = +0.140 + 0.217 + 66.411$	=	66.768
$F = +0.031 - 0.023 - 7.278$	=	— 7.270
$G = +5.292 + 16.759 + 1\,995.259 + 2.367 + 0.486 + 186.717$	=	2 206.880
$H = -2.742 + 0.079 + 19.669$	=	17.006
$I =$	=	— 241.024
$K = 0.291 + 0.104 + 58.993$	=	59.388
$L = 30.000 + 4.325 + 3.684 + 1\,140.855$	=	1 178.864
$N = -0.197 - 0.312 - 96.181$	=	— 96.690
$O = -0.350 + 0.019 + 4.586$	=	4.255
$P =$	=	54.052
$Q = +0.032 + 0.067 + 20.721$	=	— 20.820
$R = +0.006 - 0.005 - 2.484$	=	— 2.483
$S = 0.036 + 0.034 + 32.137$	=	32.207

The values of these coefficients for the right side of Arch *a* are computed as follows:

$A = 6.703 + 1.141 + 90.444 + 0.650 + 0.139 + 27.597 + (22.116) + (994.056) + (32.500)$	=	1 175.396
$B = -1.158 - 111.959 + 0.128 - 0.096 - 25.196 - (1\,241.333)$	=	— 1 379.614
$C = -0.952 - 0.184 - 36.287 - (46.562)$	=	— 83.985
$D = 12.673 + (103.444)$	=	116.117

$E = 0.140 + 0.018 + 3.690 + (3.880) = 7.728$

$F = 0.023 - 0.017 - 6.450 = -6.444$

$G = 0.456 + 16.758 + 549.958 + 0.187 + 0.485 + 172.148 + (8.866) + (1\,720.400) + (3.316) = 2\,472.574$

$H = -0.454 + 0.067 + 17.000 - (3.463) = 13.150$

$I = -116.879 - 141.867 = -258.746$

$K = 0.024 + 0.104 + 57.121 + 0.396 = 57.645$

$L = 17.910 + 1.843 + 0.254 + 49.779 + 22.166 + 3.973 + 66.708 = 162.633$

$N = -0.204 - 0.023 - 4.590 - 5.559 = -10.376$

$O = -0.066 + 0.012 + 4.125 - (0.413) = 3.658$

$P = 43.382 + 11.822 = 55.204$

$Q = 0.032 + 0.006 + 1.442 + (.463) = 1.943$

$R = 0.005 - 0.004 - 2.365 = -2.364$

$S = 0.003 + 0.034 + 31.759 + 0.047 = 31.843$

Deflections due to the post are enclosed in parentheses. To compute the deflections due to the loads, the partial summations are substituted in Equations (24), in which formula, the load is not at the crown.

For load at Point a (Fig. 3):

$L_x = -0.893 - 1\,216.792 + 0.176 - 0.131 - 22.270 + 632.376 - 0.385 + 0.287 + 54.142 = -553.490$

$L_y = 5.262 + 5.321 + 1\,744.501 + 2.366 + 0.384 + 96.572 - 956.002 - 4.271 - 0.926 - 247.271 = 645.936$

$L_z = -2.726 + 0.072 + 14.210 + 5.017 - 0.245 - 36.274 = -19.946$

$L_a = -159.334 + 90.259 = -69.075$

$L_\beta = 0.026 - 0.020 - 3.405 - 0.059 + 0.044 + 9.367 = 5.935$

$L_\gamma = 0.291 + 0.063 + 16.835 - 0.526 - 0.158 - 46.864 = -30.349$

For load at Point (b):

$R_x = -0.652 - 98.804 + 0.114 - 0.085 - 16.339 + 55.008 + 0.114 - 0.085 - 18.307 - 1\,241.331 + 620.666 = -699.431$

$R_y = 0.427 + 5.322 + 299.200 - 0.185 + 0.383 + 82.004 - 211.133 + 0.144 + 0.398 + 94.439 + 8.866 + 1\,702.399 + 3.315 - 851.200 + 7.498 = 1\,137.247$

$R_z = -0.432 + 0.060 + 11.540 - 0.368 + 0.105 + 12.681 - 3.463 - 2.609 = 17.514$

$R_a = -35.189 + 26.244 - 141.867 + 70.933 = -79.879$

$R_\beta = 0.018 - 0.013 - 2.858 + 0.019 - 0.014 - 3.275 = -6.123$

$R_\gamma = 0.023 + 0.063 + 14.962 + 0.018 + 0.068 + 17.869 + 0.396 + 0.314 = 33.713$

In the case of the crown loads, the value of the load for T_y and the load times z' for M_x may be substituted in Equations (9). With the co-ordinates, x', of Point (a) equal to 6.00, $z' = 14.688$, and with the co-ordinates, x', of Point (b) equal to 6.00, $z' = -6.312$.

From Equations (3) the deflections of the right cantilever of Arch a, due to the crown reactions and loads on Arch b are computed, as follows:

$$R_x = -(23.0 + 1\,274.184 - 123.356)\,T_{x\,B} + 2\,906.152\,T_{y\,B} - 176.728\,T_{z\,B} + 109.666\,M_{z\,B} - 6.669\,M_{y\,B} + 1\,590.167\,c + 2\,906.152\,d \quad (10a)$$

$$R_y = 1\,573.600\,T_{x\,B} + (9.2 - 3\,900.800 - 7.672)\,T_{y\,B} + 4.455\,T_{z\,B} + 147.200\,M_{z\,B} + 0.4148\,M_{x\,B} - 2\,125.030\,c - 3\,894.910\,d \quad (10b)$$

$$R_z = -176.728\,T_{x\,B} + 8.235\,T_{y\,B} - (23.0 + 5.340 - 253.192)\,T_{z\,B} - 9.554\,M_{y\,B} + 0.4452\,M_{x\,B} + 0.169\,c + 3.561\,d \quad (10c)$$

$$R_a = -131.133\,T_{x\,B} + 325.068\,T_{y\,B} - 12.267\,M_{z\,B} + 177.267\,c + 325.068\,d \quad (10d)$$

$$R_\beta = 14.727\,T_{x\,B} - 21.099\,T_{z\,B} + 0.7962\,M_{y\,B} \quad (10e)$$

and,

$$R_\gamma = -0.916\,T_{y\,B} + 0.532\,T_{z\,B} - 0.0495\,M_{x\,B} + 365.088\,c + 669.490\,d \quad (10f)$$

Equations (10) may be summed up, as follows:

$$R_x = -1\,171.927\,T_{x\,B} + 2\,906.152\,T_{y\,B} - 176.728\,T_{z\,B} + 109.666\,M_{z\,B} - 6.669\,M_{y\,B} + 1\,590.167\,c + 2\,906.152\,d \quad (11a)$$

$$R_y = 1\,573.600\,T_{x\,B} - 3\,899.265\,T_{y\,B} + 4.455\,T_{z\,B} + 147.200\,M_{z\,B} - 0.4148\,M_{x\,B} - 2\,125.030\,c - 3\,894.910\,d \quad (11b)$$

$$R_z = -176.728\,T_{x\,B} + 8.235\,T_{y\,B} - 224.995\,T_{z\,B} + 9.5544\,M_{y\,B} - 0.4452\,M_{x\,B} + 0.169\,c + 3.561\,d \quad (11c)$$

$$R_a = -131.133\,T_{x\,B} + 325.068\,T_{y\,B} - 12.267\,M_{z\,B} + 177.267\,c + 325.068\,d \quad (11d)$$

$$R_\beta = 14.727\,T_{x\,B} - 21.099\,T_{y\,B} + 0.7962\,M_{y\,B} \quad (11e)$$

and,

$$R_\gamma = -0.916\,T_{y\,B} + 0.532\,T_{z\,B} - 0.0495\,M_{x\,B} + 365.088\,c + 669.490\,d \quad (11f)$$

The deflections of the left cantilever of Arch b, due to the crown reactions of Arch a, can be written at once without further computation; thus:

$$L_x = -1\,171.927\,T_{x\,A} - 1\,573.600\,T_{y\,A} - 176.728\,T_{z\,A} - 131.133\,M_{z\,A} - 14.727\,M_{y\,A} + 786.800\,b \quad (12a)$$

$$L_y = 2\,906.152\,T_{x\,A} + 3\,899.265\,T_{y\,A} + 8.235\,T_{z\,A} + 325.068\,M_{z\,A} - 0.916\,M_{x\,A} - 1\,931.582\,b \quad (12b)$$

$$L_z = 176.728\,T_{x\,A} + 4.455\,T_{y\,A} - 224.995\,T_{z\,A} - 21.099\,M_{y\,A} - 0.532\,M_{x\,A} - 5.585\,b \quad (12c)$$

$$L_a = -109.666\,T_{x\,A} - 147.200\,T_{y\,A} - 12.267\,M_{z\,A} + 73.600\,b \quad (12d)$$

$$L_\beta = 6.669\ T_{x\,A} - 9.554\ T_{z\,A} - 0.7962\ M_{y\,A} \quad \ldots\ldots(12e)$$

$$L_\gamma = 0.4148\ T_{x\,A} + 0.4452\ T_{z\,A} - 0.0495\ M_{x\,A} + 0.5292\ b \quad \ldots\ldots(12f)$$

The equations for obtaining the crown reactions of Arch a may now be set up by placing the deflections of the left side equal to those of the right side, thus producing the following six equations:

$$2\,833.892\ T_{x\,A} + 117.162\ T_{y\,A} - 869.103\ T_{z\,A} + 225.017\ M_{z\,A} + 59.040\ M_{y\,A} - 13.714\ M_{x\,A} = 1\,171.927\ T_{x\,B} - 2\,906.152\ T_{y\,B} + 176.728\ T_{z\,B} + 109.666\ M_{z\,B} - 6.669\ M_{y\,B} + 1\,848.00\ t^\circ + 553.490\ a + 699.431\ b - 1\,590.167\ c - 2\,906.152\ d \quad \ldots\ldots(13a)$$

$$117.162\ T_{x\,A} + 4\,679.454\ T_{y\,A} + 3.856\ T_{z\,A} + 17.722\ M_{z\,A} - 13.644\ M_{y\,A} + 1.743\ M_{x\,A} = 1\,573.600\ T_{x\,B} - 3\,899.265\ T_{y\,B} + 4.455\ T_{z\,B} + 147.200\ M_{z\,B} - 0.4148\ M_{x\,B} - 645.939\ a + 1\,137.247\ b - 2\,125.030\ c - 3\,894.910\ d \quad \ldots\ldots(13b)$$

$$-869.103\ T_{x\,A} + 3.856\ T_{y\,A} + 1\,341.497\ T_{z\,A} - 86.314\ M_{y\,A} + 0.7913\ M_{x\,A} = 176.728\ T_{x\,B} - 8.235\ T_{y\,B} - 224.995\ T_{z\,B} + 9.5544\ M_{y\,B} - 0.4492\ M_{x\,B} + 1\,290.000\ t^\circ - 19.946\ a - 17.514\ b + 0.169\ c - 3.561\ d \quad \ldots\ldots(13c)$$

$$225.017\ T_{x\,A} + 17.722\ T_{y\,A} + 109.256\ M_{z\,A} = 131.133\ T_{x\,B} - 325.068\ T_{y\,B} + 12.267\ M_{z\,B} + 69.075\ a + 79.879\ b - 177.267\ c - 325.068\ d \quad \ldots\ldots(13d)$$

$$59.040\ T_{x\,A} - 13.644\ T_{y\,A} - 86.314\ T_{z\,A} + 22.763\ M_{y\,A} - 0.119\ M_{x\,A} = 14.727\ T_{x\,B} - 21.099\ T_{z\,B} + 0.7962\ M_{y\,B} - 5.953\ a - 6.123\ b \quad \ldots(13e)$$

$$-13.714\ T_{x\,A} + 1.743\ T_{y\,A} + 7.913\ T_{z\,A} - 0.119\ M_{y\,A} + 64.050\ M_{x\,A} = 0.916\ T_{y\,B} - 0.532\ T_{z\,B} + 0.0495\ M_{x\,B} + 30.349\ a - 33.713\ b - 0.019\ c + 0.396\ d \quad \ldots\ldots(13f)$$

The solution of these six equations is as follows:

$$1\,804.086\ T_{x\,A} = 982.464\ T_{x\,B} - 2\,170.911\ T_{y\,B} + 38.049\ T_{z\,B} + 81.730\ M_{z\,B} - 0.7666\ M_{y\,B} - 0.2681\ M_{x\,B} + 2\,666.278\ t^\circ + 415.954\ a + 499.621\ b - 1\,186.400\ c - 2\,168.103\ d \quad \ldots\ldots(14a)$$

$$4\,661.994\ T_{y\,A} = 1\,524.094\ T_{x\,B} - 3\,743.919\ T_{y\,B} - 24.430\ T_{z\,B} + 141.197\ M_{z\,B} + 1.1091\ M_{y\,B} - 0.4243\ M_{x\,B} - 65.639\ t^\circ - 682.886\ a + 1\,102.424\ b - 2\,038.019\ c - 3\,735.900\ d \quad \ldots\ldots(14b)$$

$$1\,013.343\ T_{z\,A} = 598.806\ T_{x\,B} - 821.460\ T_{y\,B} - 291.600\ T_{z\,B} + 30.616\ M_{z\,B} + 12.311\ M_{y\,B} - 0.5510\ M_{x\,B} + 2\,240.613\ t^\circ + 95.312\ a + 152.825\ b - 444.135\ c - 815.579\ d \quad \ldots\ldots(14c)$$

$$109.256\ M_{z\,A} = 2.800\ T_{x\,B} - 40.066\ T_{y\,B} - 4.653\ T_{z\,B} + 1.536\ M_{z\,B} + 0.0914\ M_{y\,B} + 0.0350\ M_{x\,B} - 332.306\ t^\circ + 19.791\ a + 13.372\ b - 21.545\ c - 40.447\ d \quad \ldots\ldots(14d)$$

$$22.763\ M_{y\,A} = 38.045\ T_{x\,B} - 9.897\ T_{y\,B} - 47.249\ T_{z\,B} + 0.346\ M_{z\,B} + 1.873\ M_{y\,B} - 0.0392\ M_{x\,B} + 103.406\ t^\circ - 13.385\ a - 6.289\ b - 4.978\ c - 9.464\ d \quad \ldots\ldots(14e)$$

$$64.050\ M_{x\,A} = 2.422\ T_{x\,B} - 7.823\ T_{y\,B} + 1.796\ T_{z\,B} + 0.331\ M_{z\,B} - 0.0925\ M_{y\,B} + 0.0518\ M_{x\,B} + 3.337\ t^\circ + 32.952\ a - 31.553\ b - 4.834\ c - 8.369\ d \quad \ldots\ldots(14f)$$

The labor of dividing by the coefficients of the six unknowns need not be performed if the computations are made with the aid of a machine. Substituting the values from Equations (14) in Equations (12), the loading terms on Arch b are found in terms of the crown reaction of this arch and the external loads. The substitutions are, as follows:

$$L_x = -1\,285.052\,T_{x\,B} + 2\,871.685\,T_{y\,B} + 70.539\,T_{z\,B} - 108.157\,M_{z\,B} - 3.345\,M_{y\,B} + 0.3968\,M_{x\,B} - 1\,738.666\,t^\circ - 71.419\,a - 735.296\,b + 1\,565.128\,c + 2\,866.306\,d \quad (15a)$$

$$L_y = 2\,871.685\,T_{x\,B} - 6\,754.194\,T_{y\,B} + 24.577\,T_{z\,B} + 254.566\,M_{z\,B} + 0.0659\,M_{y\,B} - 0.6878\,M_{x\,B} + 3\,269.587\,t^\circ + 158.075\,a + 1\,768.366\,b - 3\,683.369\,c - 6\,744.073\,d \quad (15b)$$

$$L_z = 70.540\,T_{x\,B} + 24.577\,T_{y\,B} - 112.229\,T_{z\,B} - 1.019\,M_{z\,B} + 4.542\,M_{y\,B} - 1.315\,M_{x\,B} + 332.239\,t^\circ - 31.064\,a - 22.155\,b + 14.902\,c + 26.030\,d \quad (15c)$$

$$L_\alpha = -108.157\,T_{x\,B} + 254.675\,T_{y\,B} - 1.020\,T_{z\,B} - 9.598\,M_{z\,B} + 0.0013\,M_{y\,B} + 0.0258\,M_{x\,B} - 122.694\,t^\circ - 5.902\,a - 66.680\,b + 138.886\,c + 254.294\,d \quad (15d)$$

$$L_\beta = -3.345\,T_{z\,B} + 0.066\,T_{y\,B} + 4.543\,T_{z\,B} + 0.001\,M_{z\,B} - 0.1844\,M_{y\,B} + 0.0056\,M_{x\,B} - 14.887\,t^\circ + 1.107\,a + 0.626\,b - 0.024\,c + 0.006\,d \quad (15e)$$

$$L_\gamma = 0.397\,T_{x\,B} - 0.687\,T_{y\,B} - 0.131\,T_{z\,B} + 0.026\,M_{z\,B} + 0.0056\,M_{y\,B} - 0.0002\,M_{x\,B} + 0.975\,t^\circ + 0.044\,a + 0.189\,b - 0.372\,c - 0.683\,d \quad (15f)$$

The coefficients in Equations (9) for Point (b) are determined in the same way as for Point (a). The computed values are as follows:

$$A = 12.500 + 1.370 + 141.887 + 7.150 + 0.753 + 153.289 + (22.167) + (1\,392.313) + (158.494) = 1\,889.923$$

$$B = -2.065 - 529.654 + 1.253 - 0.934 - 229.953 - (3\,289.533) = -4\,050.886$$

$$C = -10.437 - 0.940 - 189.669 - (227.070) = -428.116$$

$$D = 29.201 + (124.133) = 153.334$$

$$F = 0.100 - 0.075 - 21.996 = -21.971$$

$$G = 0.548 + 31.250 + 4\,499.351 + 1.010 + 5.330 + 1\,481.291 + (8.867) + (8\,302.157) + (16.185) = 14\,345.989$$

$$H = -2.490 + 0.591 + 157.823 - (9.178) = 146.746$$

$$I = -351.760 - (313.289) = -665.049$$

$$K = 0.060 + 0.486 + 173.571 + (0.874) = 174.991$$

$$L = 32.620 + 15.797 + 1.263 + 251.842 + (22.167) + (5.553) + (325.316) = 654.558$$

$$O = -0.182 + 0.050 + 15.529 - (0.496) = 14.901$$

$$P = 43.598 + (11.822) = 55.420$$

$$Q = 0.059 + 0.007 + 1.655 + (0.463) = 2.184$$

$$R = 0.009 - 0.007 - (2.547) = -2.545$$

$$S = 0.004 + 0.063 + 31.838 + (0.047) = 31.952$$

As before, deflections due to the post are enclosed in parentheses.

For load at Point (c), Fig. 31 ($x' = 12.00$ and $z' = 18.876$):

$$L_x = (-1.338 - 461.340 + 1.125 - 0.839 - 180.674 + 260.078 - 1.535 + 1.144 + 265.355 - 3\,289.533 + 1\,489.600) = -1\,917.960$$

$$L_y = (0.504 + 11.959 + 3\,044.843 + 0.997 + 4.376 + 957.615 - 1\,944.633 - 1.089 - 6.031 - 1\,425.817 + 8.866 + 8\,302.157 + 16.167 - 3\,759.467 - 16.496) = +5\,193.951$$

$$L_z = (-2.306 + 0.521 + 121.384 + 2.917 - 0.736 - 181.455 + 9.178 + 9.365) = -41.132$$

$$L_\alpha = = -225.363$$

$$L_\beta = (0.081 - 0.060 - 14.058 - 0.115 + 0.086 + 21.190) = +7.124$$

$$L_\gamma = (+0.058 + 0.320 + 75.536 - 0.065 - 0.458 - 116.602 + 0.874 - 0.892) = -41.229$$

Finally, the six equations in this case are as follows:

$$1\,889.923\,T_{xB} - 428.116\,T_{zB} + 153.334\,M_{zB} - 21.971\,M_{xB}$$
$$-1\,285.052\,T_{xB} + 70.539\,T_{zB} - 108.157\,M_{zB} + 0.397\,M_{xB} - 71.491a$$
$$604.871\,T_{xB} - 357.577\,T_{zB} + 45.177\,M_{zB} - 21.574\,M_{xB} - 71.491a$$
$$+786.800\,b - 1\,917.960\,c - 3\,820.201\,d - 2\,083.536\,t^\circ$$
$$-735.296\,b + 1\,565.128\,c + 2\,866.306\,d - 1\,738.666\,t^\circ$$
$$+51.504\,b - 352.832\,c - 953.895\,d - 3\,822.202\,t^\circ = 0 \ldots (16a)$$

$$-428.116\,T_{xB} + 654.558\,T_{zB} + 14.901\,M_{xB}$$
$$70.539\,T_{xB} - 112.229\,T_{zB} - 1.020\,M_{zB} - 0.131\,M_{xB} - 31.063\,a$$
$$357.577\,T_{xB} + 542.329\,T_{zB} - 1.020\,M_{zB} + 14.770\,M_{xB} - 31.063a$$
$$+7.812\,b - 41.132\,c - 9.714\,d - 1\,454.308\,t^\circ$$
$$-22.155\,b + 14.902\,c + 26.030\,d + 332.239\,t^\circ$$
$$-14.343\,b - 26.230\,c + 16.316\,d - 1\,122.069\,t^\circ = 0 \ldots\ldots (16c)$$

$$153.334\,T_{xB} + 55.420\,M_{zB}$$
$$-108.157\,T_{xB} - 1.020\,T_{zB} - 9.598\,M_{zB} + 0.026\,M_{xB} - 5.902a$$
$$45.177\,T_{xB} - 1.020\,T_{zB} + 45.822\,M_{zB} + 0.026\,M_{xB} - 5.902a$$
$$+73.600\,b - 225.363\,c - 665.049\,d$$
$$-66.680\,b + 138.886\,c + 254.294\,d - 122.694\,t^\circ$$
$$+6.920\,b - 86.477\,c - 410.755\,d - 122.694\,t^\circ = 0 \ldots\ldots (16d)$$

$$-21.971\ T_{x\,B} + 14.901\ T_{z\,B} + 31.952\ M_{x\,B}$$
$$0.397\ T_{x\,B} - 0.131\ T_{z\,B} + 0.026\ M_{z\,B} + 0.044a$$
$$-21.574\ T_{x\,B} + 14.770\ T_{z\,B} + 0.026\ M_{z\,B} + 31.952\ M_{x\,B} + 0.044a$$
$$-0.728\ b - 41.229\ c - 160.505\ d$$
$$+0.189\ b - 0.372\ c - 0.683\ d + 0.975\ t^\circ$$
$$-0.559\ b - 41.601\ c - 161.188\ d + 0.975\ t^\circ = 0 \quad \ldots\ldots\ldots (16f)$$

$$14\,345.989\ T_{y\,B} - 21.971\ M_{y\,B} - 1\,954.654b$$
$$-\ 6\,754.194\ T_{y\,B} + 0.066\ M_{y\,B} + 158.075\ a + 1\,768.366b$$
$$+\ 7\,591.795\ T_{y\,B} - 21.906\ M_{y\,B} + 158.075\ a - 186.288b$$
$$+5\,193.951\ c + 12\,508.596\ d$$
$$-3\,683.369\ c - 6\,744.073\ d + 3\,269.587\ t^\circ$$
$$+1\,508.657\ c + 5\,764.523\ d + 3\,269.587\ t^\circ = 0 \quad \ldots\ldots\ldots (16b)$$

$$-21.971\ T_{y\,B} + 2.184\ M_{y\,B}$$
$$+\ 0.066\ T_{y\,B} - 0.184\ M_{y\,B} + 1.107\ a + 0.626b$$
$$-21.905\ T_{y\,B} + 2.000\ M_{y\,B} + 1.107\ a + 0.626b$$
$$+7.124\ c + 4.753\ d$$
$$-0.024\ c + 0.006\ d - 14.887\ t^\circ$$
$$+7.100\ c + 4.759\ d - 14.887\ t^\circ = 0. \quad \ldots\ldots\ldots\ldots (16e)$$

In each of Equations (16), the first line of values represents the loads on Arch b itself; the second line represents the reaction from the crown of Arch a; and the final result is given in the third line. The result of solving Equations (16) is, as follows:

$$T_{x\,B} = +0.134\ a - 0.055\ b + 0.469\ c + 0.941\ d + 13.780\ t^\circ \ldots (17a)$$
$$T_{y\,B} = -0.012\ a + 0.012\ b - 0.108\ c - 0.396\ d \quad \ldots (17b)$$
$$T_{z\,B} = +0.117\ a - 0.023\ b + 0.312\ c + 0.533\ d + 11.020\ t^\circ \ldots (17c)$$
$$M_{z\,B} = -0.065\ a - 0.022\ b + 0.488\ c + 3.565\ d + 10.665\ t^\circ \ldots (17d)$$
$$M_{y\,B} = -0.404\ a - 0.023\ b - 2.958\ c - 5.522\ d \quad \ldots (17e)$$
$$M_{x\,B} = +0.036\ a - 0.018\ b + 0.823\ c + 2.908\ d + 4.188\ t^\circ \ldots (17f)$$

Substituting these values back into Equations (14) and collecting terms, the values of the crown reactions of Arch a, in terms of loads only, are:

$$T_{x\,A} = +0.317\ a + 0.231\ b - 0.242\ c - 0.039\ d + 9.697\ t^\circ \ldots . (18a)$$
$$T_{y\,A} = -0.096\ a + 0.208\ b - 0.185\ c - 0.072\ d + 4.768\ t^\circ \ldots . (18b)$$
$$T_{z\,A} = +0.142\ a + 0.114\ b - 0.185\ c - 0.042\ d + 4.974\ t^\circ \ldots . (18c)$$
$$M_{z\,A} = +0.183\ a + 0.117\ b - 0.154\ c - 0.177\ d - 3.006\ t^\circ \ldots . (18d)$$
$$M_{y\,A} = -0.635\ a - 0.329\ b - 0.272\ c - 0.178\ d + 4.860\ t^\circ \ldots . (18e)$$
$$M_{x\,A} = +0.525\ a - 0.497\ b - 0.028\ c - 0.003\ d + 0.941\ t^\circ \ldots . (18f)$$

The entire system is now broken up into a system of cantilevers which are determinate in every way, and from Equations (5) the stresses at any section can be obtained and the arch properly reinforced.

The reactions in the Y and X directions at the base of the second pier (Pier B), have been computed and found to be, respectively:

$$T_x = 0.183\,a + 0.286\,b - 0.711\,c - 0.980\,d - 0.515\,e + 0.136\,f - 0.007\,g \quad \ldots\ldots (19a)$$

and,

$$T_y = 0.085\,a + 0.816\,b + 1.077\,c + 0.677\,d + 0.139\,e + 0.012\,f - 0.018\,g \quad \ldots\ldots (19b)$$

PART IV.—An Experiment to Verify the Computations of Part III

In order to obtain an experimental verification of the theory and to check the computations given in Part III, a celluloid model was made of the structure to a scale of $\frac{3}{16}$ in. to the foot (Fig. 9). This model was tested by the deflection method devised by George E. Beggs, M. Am. Soc. C. E.,[5] and contours of deflection were obtained for the several movements of the piers. In tests applied to solid figures, Professor Beggs used very rigid apparatus.[6] The writer was not able to secure this rigidity and, therefore, used the method herein described to annul the influence of the deflections of the apparatus.

The celluloid sheet used was $1\frac{1}{8}$ in. thick, and was built up by cementing three thinner sheets together. The arch rings were cut from this sheet and the four piers were fastened to the rings with cement furnished for the purpose. The ring was cut on a milling machine. During this operation a stream of water was kept flowing on the tool to prevent fire and to keep the surface of the material cool and hard.

Although the work on the model was carefully and accurately done, it did not prove as satisfactory for the purpose as was anticipated. During the milling of the ring the sheet showed some tendency to split into the three original sheets. After several trials a ring was obtained in which this tendency did not appear to be very serious, but doubtless it had some influence on the results. The joints at which the legs were cemented to the horizontal part could not be relied upon to be 100% perfect for deflection, although they appeared to be rigid as far as could be observed. The model showed a strong tendency to warp out of shape when left out of the clamps, and doubtless there were stresses induced in it by the testing frame when the legs were in a neutral position. A line across the crown that should have been level and was made true in the milling machine, showed a crown in the center of more than $\frac{1}{20}$ in. However, even with these imperfections the writer questions whether a better model could have been made of this material.

Steel collars were secured on the model at the positions that represented the plane that was considered immovable in the bridge; that is, the contact of the ground and the foundation. As shown in Figs. 11, 12, and 13, each collar consisted of two sheets of $\frac{1}{8}$-in. steel, one on each side of a leg. These sheets were drawn toward each other by screws, thus clamping them securely

[5] This method is described in *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 1208, by Professor Beggs as applied to plane figures.

[6] *Engineering News-Record*, July 21, 1927, as applied to solid figures.

together with the leg between them. One collar was placed on each leg, so that it responded accurately to any movement of the leg at the place where it was fastened.

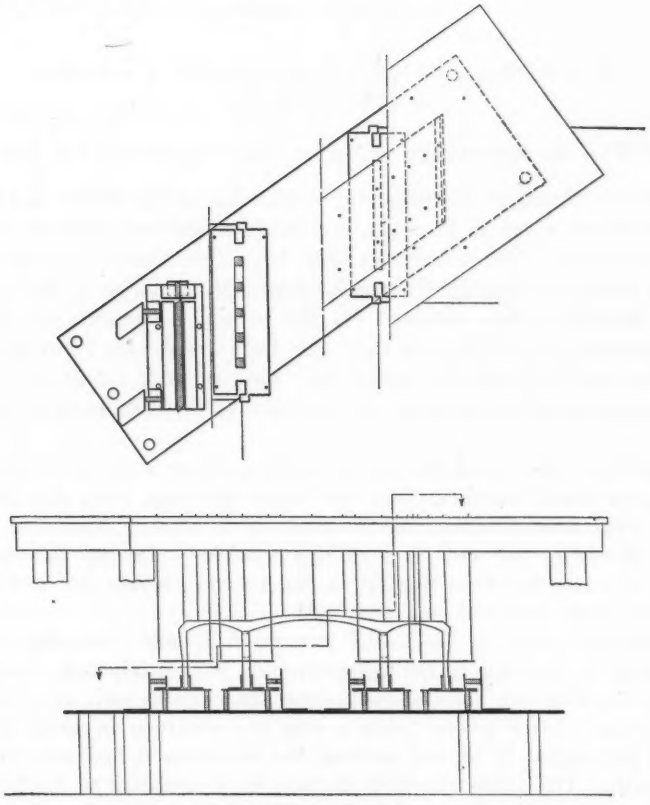


FIG. 11.—SCHEMATIC DRAWING OF TESTING APPARATUS

A horizontal plate was clamped to the bottom of each leg. Two angles, one on either side of the leg, were fastened together with bolts running through holes in the leg, thus clamping them securely to it and to each other. These angles, in turn, were bolted to the plate. This plate was approximately 2 in. below the collar. Figs. 11, 12, and 13 should serve to make clear the position and method of fastening both the collar and plate.

Each horizontal plate was fastened to a ⅜-in. steel base plate by seven bolts set in slotted holes. Four of these bolts were vertical and three, horizontal. Each horizontal bolt was passed through holes in angles that, in turn, were bolted one to the base-plate and one to the horizontal plate. Around each bolt was a strong spring, so that the horizontal plate could be moved one way by tightening the nut on a bolt, and in the opposite direction by the spring when the nut was loosened. A vertical motion could be induced by tightening or loosening all vertical bolts; a horizontal motion could be pro-

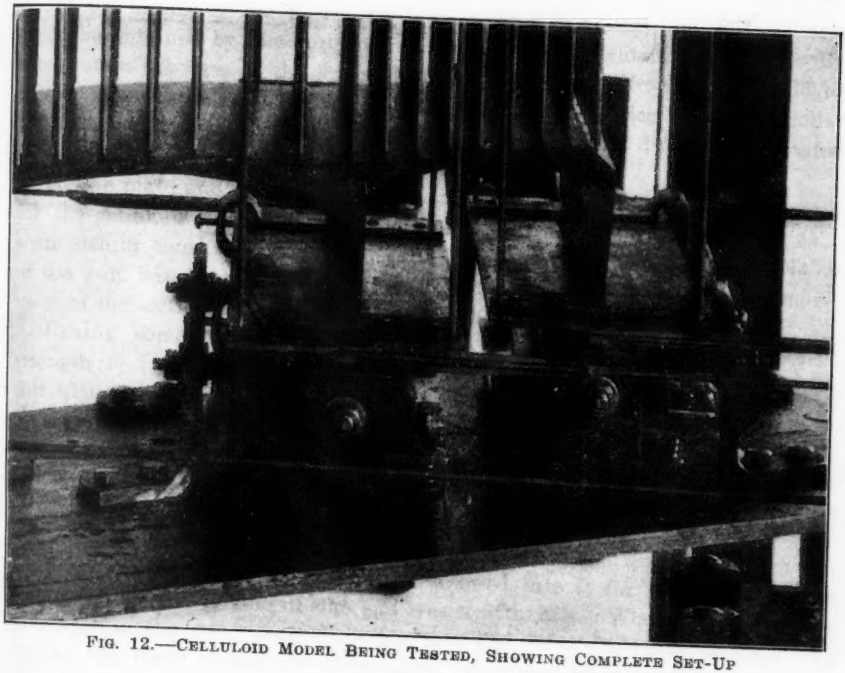


FIG. 12.—CELLULOID MODEL BEING TESTED, SHOWING COMPLETE SET-UP

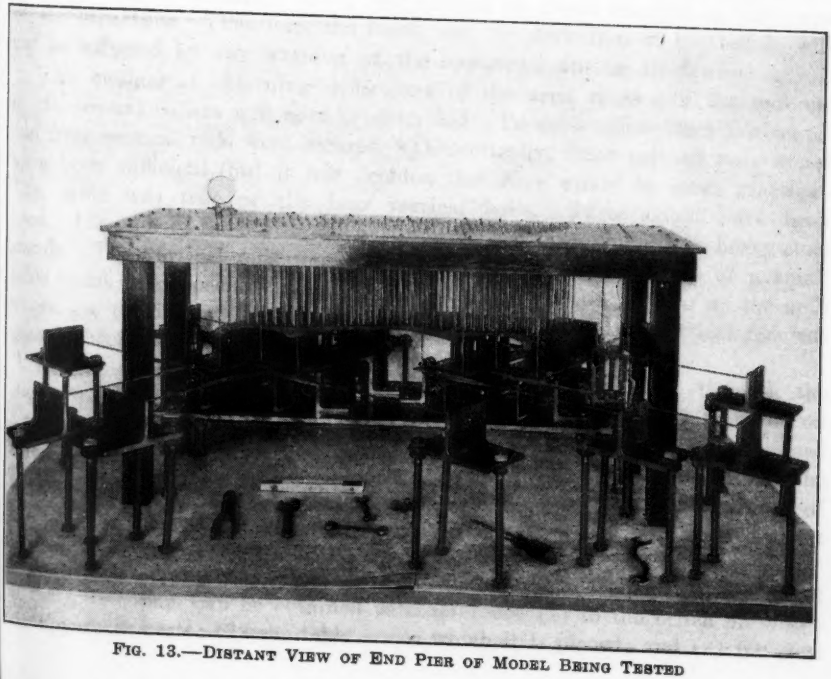


FIG. 13.—DISTANT VIEW OF END PIER OF MODEL BEING TESTED

duced by means of the horizontal bolts. Rotation about the vertical axis could be obtained by loosening one of the bolts and tightening another. By this system the base of the leg could be moved in any direction, or could be rotated about any axis. As a result the section of the leg where the collar was attached, and the collar itself, could be moved in any direction or rotated in any way.

As the plate and collar were completely independent of each other, except for the celluloid extension of the leg, any local deflection of the plate as a beam slab or cantilever, due to the force exerted by the bolts or the deflection of the $\frac{3}{8}$-in. base-plate, did not affect the collar. It only registered the movement of the section of the leg at the point where it was attached.

During some of the deformations the springs did not have sufficient strength to force the plate to the position desired. In such cases a short bolt with a nut was inserted besides the spring and used as a screw-jack by backing the nut off.

By having a collar, plate, and system of bolts on each of the four legs, any leg could be placed in any position desired with respect to reference planes or with respect to the other legs.

The $\frac{3}{8}$-in. base-plate was raised about 6 in. above the bench on which the apparatus rested by having four bolts screwed into it for legs, one in each corner. Near the middle of one end was a fifth bolt. When the model was in operation the two corner bolts on this end were shortened about $\frac{1}{4}$ in., leaving the plate resting on three legs. A three-legged structure will not have the load distribution on its legs affected by slight changes in the shape of the structure. Therefore, the loads, and the deflection of the bench, will not be affected by any warping of the apparatus during its operation.

The method of obtaining deflections of the arch rings and the motions of the several collars will next be described. To each collar three horizontal and four vertical rods were secured. Theoretically, three vertical rods would have been sufficient, but it was decided that four would be more practical. The same was true of the four vertical bolts. Three could have been used. The ends of the horizontal rods were passed through holes in heavy steel stands. The vertical rods were passed through holes in the top of a small table which was especially made for the purpose. At the position on the arch where the reading of the vertical movement was desired, a vertical rod was passed through this table top.

The amount that these rods projected above the table or through the stands was read by means of an Ames dial (reading to 0.001 in.) mounted on a tripod. This was carried from point to point by the operator and was read before and after moving one of the legs of the model. It was found that the readings were consistent by this method and no error within the limits of accuracy of the dial was found when the legs of the tripod were shifted about while the dial was on one definite rod.

With this arrangement it is to be noted that the movement of any or all of the four collars can be obtained with reference (1) to the bench on which the apparatus rests; (2) the table under which it is placed; and (3) the steel

stands. With the three-legged arrangement of the apparatus there will be no change in load on the bench, no change in deflection of the bench, and, therefore, of the stands or small table.

The position of the table with the holes in the top was adjusted by means of two plumb-bobs (see Fig. 12), the string passing through two holes in the top which were located over two punch-marks in the base-plate when the table was in proper position. The table top was of three-ply wood, the holes in it being $\frac{3}{32}$ in. larger than the rods. These holes were back-bored to $\frac{1}{4}$ in., except for the upper ply, which was about $\frac{1}{8}$ in. thick. All rods were of $\frac{1}{8}$-in. drill steel, sharpened at both ends.

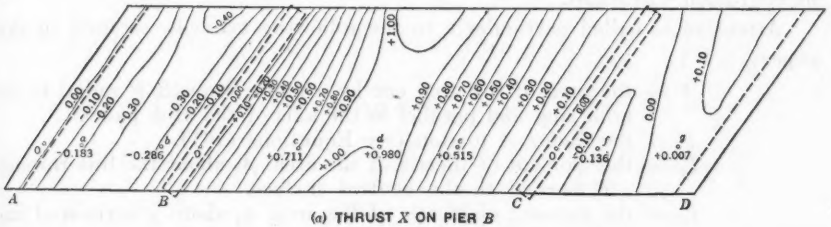
One of the horizontal rods was located on the Z-axis of the abutment. The other two were at right angles to it and equi-distant from the central point. One end of the rod rested in a punch-mark on the collar, while the other extended about $\frac{1}{8}$ in. through the steel stand which was designed to hold it. A rubber band was used to hold the rod firmly against the collar. The stands were made by sawing a 4-in. length from an 8-in. H-beam, and sawing this section in two through the web. Three holes were bored in the web, and bolts were inserted in them to form legs. On these bolts were two nuts so that the entire stand was heavy and rigid. This design allowed for adjusting the height of the stand. Holes through the web were bored $\frac{3}{32}$ in. over-size for the rods. These stands were so rigid that no deflection could be observed from a pressure such as the tripod of the Ames dial exerted. The vertical rods rested in punch-marks on the collar or on the model.

It will be noted that no effort was made to have the rods and the bolts that actuated the horizontal plate placed in line with each other, although both were placed in definite positions with respect to the axes. As a result, the moving of any bolt would affect more than one rod and if the deflection of the $\frac{3}{8}$-in. plate and other parts of the apparatus is considered, the movement of one bolt theoretically affects all the twenty-eight collar rods. However, most of these movements are very slight and with a little practice an operator could make the major adjustments, correct the rods that should not have been moved, and then go over the entire set for a closer adjustment. When the legs were finally adjusted three of them were in the same relative position with respect to the bench that they were before the adjusting was started, and the fourth had been given a definite movement either in direction or rotation as indicated by the rods on the collar of that leg. Thus, although considerable deflection had been given to the several parts of the apparatus the final position was obtained independent of these deflections.

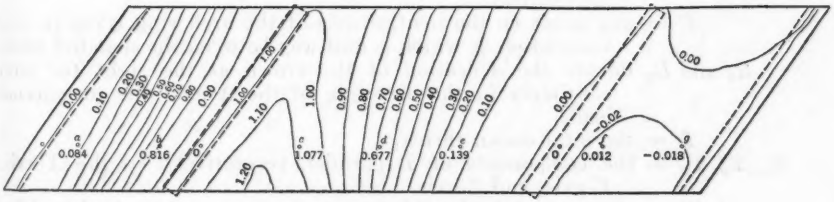
The advantage of this apparatus over the more rigid type is that it is lighter, and takes up less room, and that only one Ames dial is required to operate it.

For the square arch or the one with zero skew, the same apparatus was used. Another set of holes was bored in the base-plate for the bolts and to secure the angles; the table had a second top with holes in it for the vertical rods, and four of the steel stands had two holes in them. Except that the general manipulation was found easier, there was no other difference in the operation between the two arches.

Fig. 14(*a*) and Fig. 14(*b*) show the contour surfaces obtained for the movement of the left central pier along the X and the Y-axes (Pier B, Fig. 3). These give a general idea of the work and a comparison with the computed quantities.



(*a*) THRUST X ON PIER B



(*b*) THRUST Y ON PIER B

FIG. 14.—CONTOURS OF INFLUENCE SURFACE

ACKNOWLEDGMENT

The practical design and experimental work were conducted for the State Highway Department of Connecticut as part of a study for grade elimination between two heavily traveled highways. It is through the courtesy of E. C. Weldon, Deputy State Highway Commissioner, and that of the Commission itself, that this analysis and description are being published.

The formulas in Appendix III were derived by the writer for the Westchester County Park Commission and are reproduced through the courtesy of L. G. Holleran and Arthur G. Hayden, Members, Am. Soc. C. E.

APPENDIX I

NOTATION

As far as possible, the notation used follows the precedent established by the first paper on the single-span skew arch.[7] However, the interaction of adjacent arch spans requires the introduction of some new symbols. For example, when a factor is modified by two subscripts, the second identifies the span and the first denotes reference to a co-ordinate plane: Thus, M_{xA} refers to the

[7] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), pp. 615–616.

moment in Span A, taken about the X-axis, etc. Dimensional notation may be identified most readily by reference to the various diagrams. The capital letter coefficients introduced in Equations (9) duplicate some of the symbols used elsewhere in the paper, with other meanings. It is felt, however, that no confusion will result.

Attention is called particularly to the following symbols, as used in this paper:

A = the area of the arch cut by a plane through P radial to the arch ring and parallel to the axis of the arch panel.

F = the factor of torsion (see Equations (16)).

I_y = the moment of inertia of the area, A, about the line through P normal to the neutral surface.

I_z = the moment of inertia of the area, A, about a horizontal line through P parallel to the Z-axis.

M = the total crown moment.

M_x, M_y, M_z = the components of M about the X-axis, Y-axis, and Z-axis, respectively.

P = any point on the neutral axis of the arch ring lying in that vertical plane which is half way between the spandrel walls.

R_x and L_x denote the deflection of the crown at the right (or left) cantilever in the direction of the X-axis due to external loads.

T = the total crown thrust.

T_x, T_y, T_z = the components of T parallel, respectively, to the X-axis, Y-axis, and Z-axis.

W = the vertical load at the point x', z', or vertical dead load between the crown and P.

b = the width of the arch ring measured parallel to the Z-axis.

f = the torsional force on a block corresponding to the moment about the V-axis.

h = rise from hinge to crown on a hinged skew arch.

l = the half span of hinged skew arch.

r_A, r_B = rise of Arch a or Arch b above vertical pier.

S_A, S_B = the half span of Arch a, or Arch b, measured in X-direction.

t = the thickness of the arch measured radially at the point, P.

u, v = the co-ordinates of the crown center referred to the tangent and radial lines through P and parallel to the plane, $Z = 0$ (see Equations (1) and (2)).

x, y = the co-ordinates of P referred to vertical and horizontal axes through the crown.

x', z' = co-ordinates with respect to the crown of a concentrated load.

α, β, and γ = subscripts denoting angular rotation or moment about the Z, Y, or X-axes, respectively.

θ = angular deflection at the top of a pier due to a torsion, T, about the vertical or Y-axis.

λ = the ratio, $\dfrac{\text{modulus of elasticity in compression}}{\text{modulus of elasticity in shear}}$.

ϵ = the tangent of the angle of skew of the arch.

ϵ' = the complement of the angle between the cut section at P and the plane of the spandrels.

ϕ = the angle of slope of a plane tangent to the neutral surface of the arch at P.

w = the uniform live load per square unit, of surface.

The deflections of a skewed cantilever in the directions of the several axes and the rotations of the end of this cantilever about these axes when acted upon by loads, are given by the following equations:

$$E \Delta x = \left\{ \int \cos^2 \phi \frac{ds}{A} + \lambda \int \sin^2 \phi \frac{ds}{A} + \int y^2 \frac{ds}{I_z} + \varepsilon^2 \int x^2 \cos^2 \phi \frac{ds}{I_y} + \varepsilon^2 \lambda \int x^2 \sin^2 \phi \frac{ds}{F} \right\} T_x + \left\{ (1-\lambda) \int \sin \phi \cos \phi \frac{ds}{A} - \int x y \frac{ds}{I_z} + \varepsilon^2 \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) x^2 \sin \phi \cos \phi \, ds \right\} T_y - \left\{ \varepsilon \int v x \cos \phi \frac{ds}{I_y} + \varepsilon \lambda \int u x \sin \phi \frac{ds}{F} \right\} T_z + \left\{ \int y \frac{ds}{I_z} \right\} M_z + \left\{ \varepsilon \int x \cos^2 \phi \frac{ds}{I_y} + \varepsilon \lambda \int x \sin^2 \phi \frac{ds}{F} \right\} M_y + \left\{ \varepsilon \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) x \sin \phi \cos \phi \, ds \right\} M_x + L_x \quad (20a)$$

$$E \Delta y = \left\{ (1-\lambda) \int \sin \phi \cos \phi \frac{ds}{A} - \int x y \frac{ds}{I_z} + \varepsilon^2 \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) x^2 \sin \phi \cos \phi \, ds \right\} T_x + \left\{ \int \sin^2 \phi \frac{ds}{A} + \lambda \int \cos^2 \phi \frac{ds}{A} + \int x^2 \frac{ds}{I_z} + \varepsilon^2 \int x^2 \sin^2 \phi \frac{ds}{I_y} + \varepsilon^2 \lambda \int x^2 \cos^2 \phi \frac{ds}{F} \right\} T_y - \left\{ \varepsilon \int v x \sin \phi \frac{ds}{I_y} - \varepsilon \lambda \int u x \cos \phi \frac{ds}{F} \right\} T_z - \left\{ \int x \frac{ds}{I_z} \right\} M_z + \left\{ \varepsilon \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) x \sin \phi \cos \phi \, ds \right\} M_y + \left\{ \varepsilon \int x \sin^2 \phi \frac{ds}{I_y} + \varepsilon \lambda \int x \cos^2 \phi \frac{ds}{F} \right\} M_x + L_y \quad (20b)$$

$$E \Delta z = -\left\{ \varepsilon \int v x \cos \phi \frac{ds}{I_y} + \varepsilon \lambda \int u x \sin \phi \frac{ds}{F} \right\} T_x - \left\{ \varepsilon \int v x \sin \phi \frac{ds}{I_y} - \varepsilon \lambda \int u x \cos \phi \frac{ds}{F} \right\} T_y + \left\{ \lambda \int \frac{ds}{A} + \int v^2 \frac{ds}{I_y} + \lambda \int u^2 \frac{ds}{F} \right\} T_z - \left\{ \int v \cos \phi \frac{ds}{I_y} + \lambda \int u \sin \phi \frac{ds}{F} \right\} M_y - \left\{ \int v \sin \phi \frac{ds}{I_y} - \lambda \int u \cos \phi \frac{ds}{F} \right\} M_x + L_z \quad (20c)$$

$$E \Delta \alpha = \left\{ \int y \frac{ds}{I_z} \right\} T_x - \left\{ \int x \frac{ds}{I_z} \right\} T_y + \left\{ \int \frac{ds}{I_z} \right\} M_z + L_\alpha \quad (20d)$$

$$E \Delta \beta = \left\{ \varepsilon \int x \cos^2 \phi \frac{ds}{I_y} + \varepsilon \lambda \int x \sin^2 \phi \frac{ds}{F} \right\} T_x + \left\{ \varepsilon \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) x \sin \phi \cos \phi \, ds \right\} T_y - \left\{ \int v \cos \phi \frac{ds}{I_y} + \lambda \int u \sin \phi \frac{ds}{F} \right\} T_z + \left\{ \int \cos^2 \phi \frac{ds}{I_y} + \lambda \int \sin^2 \phi \frac{ds}{F} \right\} M_y + \left\{ \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) \sin \phi \cos \phi \, ds \right\} M_x + L_\beta \quad (20e)$$

and,

$$E\,\Delta\,\gamma = \left\{ \varepsilon \int \left(\frac{1}{I_y} - \frac{\lambda}{F}\right) x \sin\phi \cos\phi \right\} T_x + \left\{ \varepsilon \int x \sin^2\phi \frac{ds}{I_y} + \varepsilon\,\lambda \int x \cos^2\phi \frac{ds}{F} \right\} T_y - \left\{ \int v \sin\phi \frac{ds}{I_y} - \lambda \int u \cos\phi \frac{ds}{F} \right\} T_z + \left\{ \int \left(\frac{1}{I_y} - \frac{\lambda}{F}\right) \sin\phi \cos\phi\, ds \right\} M_y + \left\{ \int \sin^2\phi \frac{ds}{I_y} + \lambda \int \cos^2\phi \frac{ds}{F} \right\} M_x + L_\gamma \quad \ldots\ldots (20f)$$

This set of equations may be written as follows, if finite blocks are used:

$$E\,\Delta\,x = \left\{ \sum \cos^2\phi \frac{S}{A} + \lambda \sum \sin^2\phi \frac{S}{A} + \sum y^2 \frac{S}{I_z} + \varepsilon^2 \sum x^2 \cos^2\phi \frac{S}{I_y} + \varepsilon^2\,\lambda \sum x^2 \sin^2\phi \frac{S}{F} \right\} T_x + \left\{ (1-\lambda) \sum \sin\phi \cos\phi \frac{S}{A} - \sum x\,y \frac{S}{I_z} + \varepsilon^2 \sum x^2 \sin\phi \cos\phi \frac{S}{I_y} - \varepsilon^2 \sum x^2 \sin\phi \cos\phi \frac{S}{F} \right\} T_y - \left\{ \varepsilon \sum v\,x \cos\phi \frac{S}{I_y} + \varepsilon\,\lambda \sum u\,x \sin\phi \frac{S}{F} \right\} T_z + \left\{ \sum y \frac{S}{I_z} \right\} M_z + \left\{ \varepsilon \sum x \cos^2\phi \frac{S}{I_y} + \varepsilon\lambda \sum x \sin^2\phi \frac{S}{F} \right\} M_y + \left\{ \varepsilon \sum x \sin\phi \cos\phi \frac{S}{I_y} - \varepsilon\,\lambda \sum x \sin\phi \cos\phi \frac{S}{F} \right\} M_x + L_x \quad \ldots\ldots (21a)$$

$$E\,\Delta\,y = \left\{ (1-\lambda) \sum \sin\phi \cos\phi \frac{S}{A} - \sum x\,y \frac{S}{I_z} + \varepsilon^2 \sum x^2 \sin\phi \cos\phi \frac{S}{I_y} - \varepsilon^2\lambda \sum x^2 \sin\phi \cos\phi \frac{S}{F} \right\} T_x + \left\{ \sum \sin^2\phi \frac{S}{A} + \lambda \sum \cos^2\phi \frac{S}{A} + \sum x^2 \frac{S}{I_z} + \varepsilon^2 \sum x^2 \sin^2\phi \frac{S}{I_y} + \varepsilon^2\,\lambda \sum x^2 \cos^2\phi \frac{S}{F} \right\} T_y - \left\{ \varepsilon \sum v\,x \sin\phi \frac{S}{I_y} - \varepsilon\,\lambda \sum u\,x \cos\phi \frac{S}{F} \right\} T_z - \left\{ \sum x \frac{S}{I_z} \right\} M_z + \left\{ \varepsilon \sum x \sin\phi \cos\phi \frac{S}{I_y} - \varepsilon\,\lambda \sum x \sin\phi \cos\phi \frac{S}{F} \right\} M_y + \left\{ \varepsilon \sum x \sin^2\phi \frac{S}{I_y} + \varepsilon\,\lambda \sum x \cos^2\phi \frac{S}{F} \right\} M_x + L_y \quad \ldots\ldots (21b)$$

$$E\,\Delta\,z = -\left\{ \varepsilon \sum v\,x \cos\phi \frac{S}{I_y} + \varepsilon\,\lambda \sum u\,x \sin\phi \frac{S}{F} \right\} T_x - \left\{ \varepsilon \sum v\,x \sin\phi \frac{S}{I_y} - \varepsilon\,\lambda \sum u\,x \cos\phi \frac{S}{F} \right\} T_y + \left\{ \lambda \sum \frac{S}{A} + \sum v^2 \frac{S}{I_y} + \lambda \sum u^2 \frac{S}{F} \right\} T_z - \left\{ \sum v \cos\phi \frac{S}{I_y} + \lambda \sum u \sin\phi \frac{S}{F} \right\} M_y - \left\{ \sum v \sin\phi \frac{S}{I_y} - \lambda \sum u \cos\phi \frac{S}{F} \right\} M_x + L_z \quad \ldots\ldots (21c)$$

$$E\,\Delta\,\alpha = \left\{ \sum y \frac{S}{I_z} \right\} T_x - \left\{ \sum x \frac{S}{I_z} \right\} T_y + \left\{ \sum \frac{S}{I_z} \right\} M_z + L_\alpha \quad \ldots (21d)$$

$$E\,\Delta\,\beta = \left\{ \varepsilon \sum x \cos^2\phi \frac{S}{I_y} + \varepsilon\,\lambda \sum x \sin^2\phi \frac{S}{F} \right\} T_x + \left\{ \varepsilon \sum x \sin\phi\cos\phi \frac{S}{I_y} - \varepsilon\,\lambda \sum x \sin\phi\cos\phi \frac{S}{F} \right\} T_y - \left\{ \sum v \cos\phi \frac{S}{I_y} + \lambda \sum u \sin\phi \frac{S}{F} \right\} T_z + \left\{ \sum \cos^2\phi \frac{S}{I_y} + \lambda \sum \sin^2\phi \frac{S}{F} \right\} M_y + \left\{ \sum \sin\phi\cos\phi \frac{S}{I_y} - \lambda \sum \sin\phi\cos\phi \frac{S}{F} \right\} M_x + L_\beta \quad \ldots\ldots\ldots\ldots\ldots (21e)$$

and,

$$E\,\Delta\,\gamma = \left\{ \varepsilon \sum x \sin\phi\cos\phi \frac{S}{I_y} - \varepsilon\,\lambda \sum x \sin\phi\cos\phi \frac{S}{F} \right\} T_x + \left\{ \varepsilon \sum x \sin^2\phi \frac{S}{I_y} + \varepsilon\,\lambda \sum x \cos^2\phi \frac{S}{F} \right\} T_y - \left\{ \sum v \sin\phi \frac{S}{I_y} - \lambda \sum u \cos\phi \frac{S}{F} \right\} T_z + \left\{ \sum \sin\phi\cos\phi \frac{S}{I_y} - \lambda \sum \sin\phi\cos\phi \frac{S}{F} \right\} M_y + \left\{ \sum \sin^2\phi \frac{S}{I_y} + \lambda \sum \cos^2\phi \frac{S}{F} \right\} M_z + L_\gamma \quad \ldots\ldots\ldots (21f)$$

In the usual type of arch the terms due to shear and direct thrust may be omitted; this is also true for those due to rotation about the radial axis if the arch is of the barrel type:

$$E\,\Delta\,x = \left\{ \sum y^2 \frac{S}{I_z} + \varepsilon^2\,\lambda \sum x^2 \sin^2\phi \frac{S}{F} \right\} T_x + \left\{ -\sum x\,y \frac{S}{I_z} - \varepsilon^2\,\lambda \sum x^2 \sin\phi\cos\phi \frac{S}{F} \right\} T_y - \left\{ \varepsilon\,\lambda \sum u\,x \sin\phi \frac{S}{F} \right\} T_z + \left\{ \sum y \frac{S}{I_z} \right\} M_z + \left\{ \varepsilon\,\lambda \sum x \sin^2\phi \frac{S}{F} \right\} M_y - \left\{ \varepsilon\,\lambda \sum x \sin\phi\cos\phi\, S \right\} M_x + L_x \quad \ldots\ldots\ldots\ldots (22a)$$

$$E\,\Delta\,y = \left\{ -\sum x\,y \frac{S}{I_z} - \varepsilon^2\,\lambda \sum x^2 \sin\phi\cos\phi \frac{S}{F} \right\} T_x + \left\{ \sum x^2 \frac{S}{I_z} + \varepsilon^2\,\lambda \sum x^2 \cos^2\phi \frac{S}{F} \right\} T_y + \left\{ \varepsilon\,\lambda \sum u\,x \cos\phi \frac{S}{F} \right\} T_z - \left\{ \sum x \frac{S}{I_z} \right\} M_z - \left\{ \varepsilon\,\lambda \sum x \sin\phi\cos\phi \frac{S}{F} \right\} M_y + \left\{ \varepsilon\,\lambda \sum x \cos^2\phi \frac{S}{F} \right\} M_x + L_y \quad \ldots (22b)$$

$$E\,\Delta\,z = -\left\{ \varepsilon\,\lambda \sum u\,x \sin\phi \frac{S}{F} \right\} T_x + \left\{ \varepsilon\,\lambda \sum u\,x \cos\phi \frac{S}{F} \right\} T_y + \left\{ \lambda \sum u^2 \frac{S}{F} \right\} T_z - \left\{ \lambda \sum u \sin\phi \frac{S}{F} \right\} M_y + \left\{ \lambda \sum u \cos\phi \frac{S}{F} \right\} M_x + L_z \quad \ldots\ldots\ldots\ldots\ldots (22c)$$

$$E\,\Delta\,\alpha = \left\{ \sum y \frac{S}{I_z} \right\} T_x - \left\{ \sum x \frac{S}{I_z} \right\} T_y + \left\{ \sum \frac{S}{I_z} \right\} + L_\alpha \quad \ldots\ldots (22d)$$

$$E \Delta \beta = \left\{ \varepsilon \lambda \sum x \sin^2 \phi \frac{S}{F} \right\} T_x - \left\{ \varepsilon \lambda \sum x \sin \phi \cos \phi \frac{S}{F} \right\} T_y + \left\{ \lambda \sum u \sin \phi \frac{S}{F} \right\} T_z + \left\{ \lambda \sum \sin^2 \phi \frac{S}{F} \right\} M_y - \left\{ \lambda \sum \sin \phi \cos \phi \frac{S}{F} \right\} M_x + L_\beta \quad (22e)$$

and,

$$E \Delta \gamma = - \left\{ \varepsilon \lambda \sum x \sin \phi \cos \phi \frac{S}{F} \right\} T_x + \left\{ \varepsilon \lambda \sum x \cos^2 \phi \frac{S}{F} \right\} T_y + \left\{ \lambda \sum u \cos \phi \frac{S}{F} \right\} T_z - \left\{ \lambda \sum \sin \phi \cos \phi \frac{S}{F} \right\} M_y + \left\{ \lambda \sum \cos^2 \phi \frac{S}{F} \right\} M_x + L_\gamma \quad (22f)$$

Terms for various types of load have been developed, as follows:

Temperature Loading Terms:

$$L_x = S_A \, t^\circ \, e \, E \quad (23a)$$

$$L_z = \varepsilon \, S_A \, t^\circ \, e \, E \quad (23b)$$

Concentrated Loading Terms (the integrations or summations of Equations (24) and (25) are taken from the load to the abutment only):

$$L_x = \left\{ (1 - \lambda) \int \sin \phi \cos \phi \frac{ds}{A} - \int x \, y \frac{ds}{I_z} + \varepsilon^2 \int x^2 \sin \phi \cos \phi \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) ds \right\} W + \left\{ \int y \frac{ds}{I_z} \right\} W x' - \left\{ \varepsilon \int x \sin \phi \cos \phi \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) ds \right\} W z' \quad (24a)$$

$$L_y = \left\{ \int \sin^2 \phi \frac{ds}{A} + \lambda \int \cos^2 \phi \frac{ds}{A} + \int x^2 \frac{ds}{I_z} + \varepsilon^2 \int x^2 \sin^2 \phi \frac{ds}{I_y} + \varepsilon^2 \lambda \int x^2 \cos^2 \phi \frac{ds}{F} \right\} W - \left\{ \int x \frac{ds}{I_z} \right\} W x' - \left\{ \varepsilon \int x \sin^2 \phi \frac{ds}{I_y} + \varepsilon \lambda \int x \cos^2 \phi \frac{ds}{F} \right\} W z' \quad (24b)$$

$$L_z = - \left\{ \varepsilon \int v \, x \sin \phi \frac{ds}{I_y} - \varepsilon \lambda \int u \, x \cos \phi \frac{ds}{F} \right\} W + \left\{ \int v \sin \phi \frac{ds}{I_y} - \lambda \int u \cos \phi \frac{ds}{F} \right\} W z' \quad (24c)$$

$$L_\alpha = - \left\{ \int x \frac{ds}{I_z} \right\} W + \left\{ \int \frac{ds}{I_z} \right\} W x' \quad (24d)$$

$$L_\beta = \left\{ \varepsilon \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) x \sin \phi \cos \phi \, ds \right\} W - \left\{ \int \left(\frac{1}{I_y} - \frac{\lambda}{F} \right) \sin \phi \cos \phi \, ds \right\} W z' \quad (24e)$$

and,

$$L_\gamma = \left\{ \varepsilon \int x \sin^2 \phi \frac{ds}{I_y} + \varepsilon \lambda \int x \cos^2 \phi \frac{ds}{F} \right\} W - \left\{ \int \sin^2 \phi \frac{ds}{I_y} + \lambda \int \cos^2 \phi \frac{ds}{F} \right\} W z' \quad \ldots\ldots(24f)$$

As in the case of Equations (21) the terms of Equations (24) may be written for finite blocks, as follows:

Concentrated Loading Terms:

$$L_x = -\left\{ \sum x y \frac{S}{I_z} + \varepsilon^2 \sum x^2 \sin \phi \cos \phi \frac{\lambda}{F} S \right\} W + \sum y \frac{S}{I_z} W x' + \varepsilon \sum x \sin \phi \cos \phi \frac{\lambda s}{F} W z' \quad \ldots\ldots(25a)$$

$$L_y = \left\{ \sum x^2 \frac{S}{I_z} + \varepsilon^2 \lambda \sum x^2 \cos^2 \phi \frac{S}{F} \right\} W - \left\{ \sum x \frac{S}{I_z} W x' - \varepsilon \lambda \sum x \cos^2 \phi \frac{S}{F} \right\} W z' \quad \ldots\ldots(25b)$$

$$L_z = \left\{ \varepsilon \lambda \sum u x \cos \phi \frac{S}{F} \right\} W - \left\{ \lambda \sum u \cos \phi \frac{S}{F} \right\} W z' \quad \ldots\ldots(25c)$$

$$L_\alpha = -\left\{ \sum x \frac{S}{I_z} \right\} W + \left\{ \sum \frac{S}{I_z} \right\} W x' \quad \ldots\ldots(25d)$$

$$L_\beta = -\left\{ \varepsilon \sum x \sin \phi \cos \phi \frac{\lambda}{F} S \right\} W + \left\{ \sum \sin \phi \cos \phi \frac{\lambda}{F} S \right\} W z' \quad \ldots\ldots(25e)$$

and,

$$L_\gamma = \left\{ \varepsilon \lambda \sum x \cos^2 \phi \frac{S}{F} \right\} W - \left\{ \lambda \sum \cos^2 \phi \frac{S}{F} \right\} W z' \quad \ldots\ldots(25f)$$

In the case of dead load and uniform load the terms due to horizontal earth thrust have been omitted for the sake of brevity. The remaining terms, have been reproduced for the finite blocks only:

Dead Vertical Loading Terms:

$$L_x = \sum (1-\lambda) \sum W \sin \phi \cos \phi \frac{S}{A} - \sum W \bar{x} y \frac{S}{I_z} + \varepsilon^2 \sum W \bar{x} x \sin \phi \cos \phi S - \varepsilon^2 \lambda \sum W \bar{x} x \sin \phi \cos \phi \frac{S}{F} \quad \ldots\ldots(26a)$$

$$L_y = \sum W \sin^2 \phi \frac{S}{A} + \lambda \sum W \cos^2 \phi \frac{S}{A} + \sum W \bar{x} x \frac{S}{I_z} + \varepsilon^2 \sum W \bar{x} x \sin^2 \phi \frac{S}{I_y} + \varepsilon^2 \lambda \sum W \bar{x} x \cos^2 \phi \frac{S}{F} \quad \ldots\ldots(26b)$$

$$L_z = -\varepsilon \sum W \bar{x} v \sin \phi \frac{S}{I_y} + \varepsilon \lambda \sum W \bar{x} u \cos \phi \frac{S}{F} \quad \ldots\ldots(26c)$$

$$L_\alpha = -\sum W \bar{x} \frac{S}{I_z} \quad \ldots\ldots(26d)$$

$$L_\beta = \varepsilon \sum W \bar{x} \sin \phi \cos \phi \frac{S}{I_y} - \varepsilon \lambda \sum W \bar{x} \sin \phi \cos \phi \frac{S}{F} \quad \ldots\ldots (26e)$$

and,

$$L_\gamma = \varepsilon \sum W \bar{x} \sin^2 \phi \frac{S}{I_y} + \lambda \varepsilon \sum W \bar{x} \cos^2 \phi \frac{S}{F} \quad \ldots\ldots (26f)$$

Uniform Vertical Loading Terms:

$$L_x = b w \left\{ (1 - \lambda) \sum x \sin \phi \cos \phi \frac{S}{A} - \frac{1}{2} \sum x^2 y \frac{S}{I_z} + \frac{1}{2} \varepsilon^2 \sum x^3 \sin \phi \cos \phi \frac{S}{I_y} - \frac{1}{2} \varepsilon^2 \lambda \sum x^3 \sin \phi \cos \phi \frac{S}{F} \right\} \quad \ldots (27a)$$

$$L_y = b w \left\{ \sum x \sin^2 \phi \frac{S}{A} + \lambda \sum x \cos^2 \phi \frac{S}{A} + \frac{1}{2} \sum x^3 \frac{S}{I_z} + \frac{1}{2} \varepsilon^2 \sum x^3 \sin^2 \phi \frac{S}{I_y} + \frac{1}{2} \lambda \varepsilon^2 \sum x^3 \cos^2 \phi \frac{S}{F} \right\} \quad \ldots\ldots (27b)$$

$$L_z = b w \left\{ - \frac{1}{2} \varepsilon \sum x^2 v \sin \phi \frac{S}{I_y} + \frac{1}{2} \lambda \varepsilon \sum x^2 u \cos \phi \frac{S}{F} \right\} \quad \ldots (27c)$$

$$L_\alpha = b w \left\{ - \frac{1}{2} \sum x^2 \frac{S}{I_z} \right\} \quad \ldots\ldots (27d)$$

$$L_\beta = b w \left\{ \frac{1}{2} \varepsilon \sum x^2 \sin \phi \cos \phi \frac{S}{I_y} - \frac{1}{2} \varepsilon \lambda \sum x^2 \sin \phi \cos \phi \frac{S}{F} \right\} \quad \ldots (27e)$$

and,

$$L_\gamma = b w \left\{ \frac{1}{2} \varepsilon \sum x^2 \sin^2 \phi \frac{S}{I_y} + \frac{1}{2} \lambda \varepsilon \sum x^2 \cos^2 \phi \frac{S}{F} \right\} \quad \ldots\ldots (27f)$$

The value of F is given by the expression:

$$\frac{I}{F} = \frac{205 \pi (t^2 + b^2)}{180 b^3 t^3} \quad \ldots\ldots (28)$$

When λ is 2.5 this becomes $\frac{\lambda}{F} = 8.95 \frac{1}{b} \left(\frac{1}{b^2 t} + \frac{1}{t^3} \right)$, which may be written:

$$\frac{\lambda}{F} = 9 \times \frac{1}{b} \left(\frac{1}{b^2 t} + \frac{1}{t^3} \right) \quad \ldots\ldots (29)$$

If the angle of skew is made equal to zero, these equations reduce as follows:

Equations (20) become,

$$E \Delta x = \left\{ \int \cos^2 \phi \frac{ds}{A} + \lambda \int \sin^2 \phi \frac{ds}{A} + \int y^2 \frac{ds}{I_z} \right\} T_x + \left\{ (1 - \lambda) \int \sin \phi \cos \phi \frac{ds}{A} - \int x y \frac{ds}{I_z} \right\} T_y + \left\{ \int y \frac{ds}{I_z} \right\} M_z + L_x \quad \ldots (30a)$$

$$E\,\Delta y = \left\{(1-\lambda)\int \sin\phi\cos\phi\,\frac{ds}{A} - \int x\,y\,\frac{ds}{I_z}\right\} T_x + \left\{\int \sin^2\phi\,\frac{ds}{A} + \lambda\int \cos^2\phi\,\frac{ds}{A} + \int x^2\,\frac{ds}{I_z}\right\} T_y + \left\{\int x\,\frac{ds}{I_z}\right\} M_z + L_y \quad (30b)$$

$$E\,\Delta\alpha = \left\{\int \gamma\frac{ds}{I_z}\right\} T_x - \left\{\int x\frac{ds}{I_z}\right\} T_y + \left\{\int \frac{ds}{I_z}\right\} M_z + L_\alpha \quad (30d)$$

$$E\,\Delta z = \left\{\lambda\int \frac{ds}{A} + \int v^2\frac{ds}{I_y} + \lambda\int u^2\,\frac{ds}{F}\right\} T_z - \left\{\int v\cos\phi\,\frac{ds}{I_y} + \lambda\int u\sin\phi\,\frac{ds}{F}\right\} M_y - \left\{\int v\sin\phi\,\frac{ds}{I_y} - \lambda\int u\cos\phi\,\frac{ds}{F}\right\} M_x + L_z \quad (30c)$$

$$E\,\Delta\beta = -\left\{\int v\cos\phi\,\frac{ds}{I_y} + \lambda\int u\sin\phi\,\frac{ds}{F}\right\} T_z + \left\{\int \cos^2\phi\,\frac{ds}{I_y} + \lambda\int \sin^2\phi\,\frac{ds}{F}\right\} M_y + \left\{\int\left(\frac{1}{I_y} - \frac{\lambda}{F}\right)\sin\phi\cos\phi\,ds\right\} M_x + L_\beta \quad (30e)$$

and,

$$E\,\Delta\gamma = -\left\{\int v\sin\phi\,\frac{ds}{I_y} - \lambda\int u\cos\phi\,\frac{ds}{F}\right\} T_z + \left\{\int\left(\frac{1}{I_y} - \frac{\lambda}{F}\right)\sin\phi\cos\phi\,ds\right\} M_y + \left\{\int \sin^2\phi\,\frac{ds}{I_y} + \lambda\int \cos^2\phi\,\frac{ds}{F}\right\} M_x + L_\gamma \quad (30f)$$

Equations (24) become

$$L_x = \left\{(1-\lambda)\int \sin\phi\cos\phi\,\frac{ds}{A} - \int x\,y\,\frac{ds}{I_z}\right\} W + \left\{\int y\,\frac{ds}{I_z}\right\} W x' \quad (31a)$$

$$L_y = \left\{\int \sin^2\phi\,\frac{ds}{A} + \lambda\int \cos^2\phi\,\frac{ds}{A} + \int x^2\,\frac{ds}{I_z}\right\} W - \left\{\int x\,\frac{ds}{I_z}\right\} W x' \quad (31b)$$

$$L_\alpha = -\left\{\int x\,\frac{ds}{I_z}\right\} W + \left\{\int \frac{ds}{I_z}\right\} W x' \quad (31d)$$

$$L_z = -\left\{\int v\sin\phi\,\frac{ds}{I_y} - \lambda\int u\cos\phi\,\frac{ds}{F}\right\} W z' \quad (31c)$$

$$L_\beta = -\left\{\int\left(\frac{1}{I_y} - \frac{\lambda}{F}\right)\sin\phi\cos\phi\,ds\right\} W z' \quad (31e)$$

and,

$$L_\gamma = -\left\{ \int \sin^2 \phi \frac{ds}{I_y} + \lambda \int \cos^2 \phi \frac{ds}{F} \right\} W z' \quad \ldots\ldots (31f)$$

Equations (3) become,

$$R_x = \left\{ \int \frac{\lambda}{A} dy + \int (y + r_A)(y + r_B) \frac{dy}{I_y} \right\} T_{x\,B} + \left\{ s_B \int (y + r_A) \frac{dy}{I_z} \right\} T_{y\,B} - \left\{ (y + r_A) \frac{dy}{I_z} \right\} M_{z\,B} + \left\{ s_B \int (y + r_A) \frac{dy}{I_z} \right\} \sum L - \left\{ (y + r_A) \frac{dy}{I_z} \right\} \sum L\, x' \quad \ldots (32a)$$

$$R_y = \left\{ s_A \int (y + r_A) \frac{dy}{I_z} \right\} T_{x\,B} + \left\{ \int \frac{dy}{A} - s_A\, s_B \int \frac{dy}{I_z} \right\} T_{y\,B} + \left\{ s_A \int \frac{dy}{I_z} \right\} M_{z\,B} + \left\{ \int \frac{dy}{A} - s_A\, s_B \int \frac{dy}{I_z} \right\} \sum L + \left\{ s_A \int \frac{dy}{I_y} \right\} \sum L\, z' + \left\{ s_A \int \frac{dy}{I_z} \right\} \sum L\, x' \quad \ldots\ldots (32b)$$

$$R_\alpha = -\left\{ \int (y + r_B) \frac{dy}{I_z} \right\} T_{x\,B} + s_B \left\{ \int \frac{dy}{I_z} \right\} T_{y\,B} - \left\{ \int \frac{dy}{I_z} \right\} M_{z\,B} + \left\{ s_B \int \frac{dy}{I_z} \right\} \sum L - \left\{ \int \frac{dy}{I_z} \right\} \sum L\, x' \quad \ldots\ldots (32d)$$

$$R_z = -\left\{ \lambda \int \frac{dy}{A} + \int (y + r_A)(y + r_B) \frac{dy}{I_y} - s_A\, s_B\, \lambda \int \frac{dy}{F} \right\} T_z - \left\{ s_A\, \lambda \int \frac{dy}{F} \right\} M_{y\,B} + \left\{ \int (y + r_A) \frac{dy}{I_y} \right\} M_{x\,B} - \left\{ \int (y + r_A) \frac{dy}{I_y} \right\} \sum L\, z' \quad \ldots\ldots (32c)$$

$$R_\beta = -\left\{ s_B\, \lambda \int \frac{dy}{F} \right\} T_{z\,B} + \left\{ \lambda \int \frac{dy}{F} \right\} M_{y\,B} \quad \ldots\ldots (32e)$$

and,

$$R_\gamma = \left\{ (y + r_B) \int \frac{dy}{I_y} \right\} T_z - \left\{ \int \frac{dy}{I_y} \right\} M_x + \left\{ \int \frac{dy}{I_y} \right\} \sum L\, z' \quad \ldots (32f)$$

and Equations (4) become,

$$L_x = -\left\{ \int \frac{\lambda}{A} dy + \int (y + r_A)(y + r_B) \frac{dy}{I_z} \right\} T_{x\,A} - \left\{ s_A \int (y + r_B) \frac{dy}{I_z} \right\} T_{y\,A} - \left\{ \int (y + r_B) \frac{dy}{I_z} \right\} M_{z\,A} + \left\{ s_A \int (y + r_B) \frac{dy}{I_z} \right\} \sum L - \left\{ \int (y + r_B) \frac{dy}{I_z} \right\} \sum L\, x' \quad \ldots (33a)$$

$$L_y = \left\{ s_B \int (y + r_A) \frac{dy}{I_z} \right\} T_{x\,A} + \left\{ s_A\ s_B \int \frac{dy}{I_z} - \int \frac{dy}{A} \right\} T_{y\,A}$$
$$+ \left\{ s_B \int \frac{dy}{I_z} \right\} M_{z\,A}$$
$$- \left\{ s_A\ s_B \int \frac{dy}{I_z} - \int \frac{dy}{A} + s_A\ s_B \int \frac{dy}{I_y} \right\} \sum L$$
$$+ \left\{ s_B \int \frac{dy}{I_z} \right\} \sum L\,x' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33b)$$

$$L_a = - \left\{ \int (y + r_A) \frac{dy}{I_z} \right\} T_{x\,A} - \left\{ s_A \int \frac{dy}{I_z} \right\} T_{y\,A} - \left\{ \int \frac{dy}{I_z} \right\} M_{z\,A}$$
$$+ \left\{ s_A \int \frac{dy}{I_z} \right\} \sum L - \left\{ \int \frac{dy}{I_z} \right\} \sum L\,x' \quad \ldots\ldots\ldots\ldots (33d)$$

$$L_z = \left\{ s_A\ s_B\ \lambda \int \frac{dy}{F} - \lambda \int \frac{dy}{A} - \int (y + r_A)\,(y + r_B) \frac{dy}{I_y} \right\} T_{z\,A}$$
$$+ \left\{ s_B\ \lambda \int \frac{dy}{F} \right\} M_{y\,A} + \left\{ \int (y + r_B) \frac{dy}{I_y} \right\} M_{x\,A}$$
$$- \left\{ \int (y + r_B) \frac{dy}{I_y} \right\} \sum L\,z' \quad \ldots\ldots\ldots\ldots\ldots\ldots (33c)$$

$$L_\beta = - \left\{ s_A\ \lambda \int \frac{dy}{F} \right\} T_{z\,A} - \left\{ \lambda \int \frac{dy}{F} \right\} M_{y\,A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33e)$$

and,

$$L_\gamma = \left\{ \int (y + r_B) \frac{dy}{I_y} \right\} T_{z\,A} - \left\{ \int \frac{dy}{I_y} \right\} M_{x\,A} - \left\{ \int \frac{dy}{I_y} \right\} \sum L\,z' \quad (33f)$$

All groups of equations for zero skew are arranged in two parts of three equations each, so that two sets of computations can be carried through entirely independent of each other.

APPENDIX III

Equations for Skew Arch, When Abutments Are Free to Rotate

As the formulas for the skew arch, when the abutments are considered free to rotate, have been requested on numerous occasions they are given in Appendix III with a brief outline of their derivation.

From Fig. 3 and Equations (5),

$$dv = t_v \frac{ds}{A\ E} = (T_x \cos \phi \pm T_y \sin \phi + V \sin \phi - H \cos \phi) \frac{ds}{A\ E} \quad \ldots\ldots (34a)$$

$$du = t_u \frac{\lambda\ ds}{A\ E} = (- T_x \sin \phi \pm T_y \cos \phi + V \cos \phi + H \sin \phi) \frac{\lambda\ ds}{A\ E} \quad \ldots (34b)$$

$$dz = t_z \frac{\lambda\ ds}{A\ E} = T_z \frac{\lambda\ ds}{A\ E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34c)$$

$$d\alpha = m_z \frac{ds}{E\,I_z} = [T_x\,y \mp T_y\,x + M_z - V\,(x - x') - H\,(y - y')]\,\frac{ds}{E\,I_z} \quad \ldots(34d)$$

$$d\beta = m_u \frac{ds}{E\,I_y} = [T_x\,\varepsilon\,x\cos\phi \pm T_y\,x\sin\phi - T_z\,v \pm M_y\cos\phi + M_x\sin\phi + V\,(\varepsilon\,x - z')\sin\phi - H\,(\varepsilon\,x - z')\cos\phi]\,v \quad \ldots(34e)$$

$$d\gamma = m_v \frac{\lambda\,ds}{E\,F} = [-T_x\,\varepsilon\,x\sin\phi \pm T_y\,x\cos\phi + T_z\,u \mp M_y\sin\phi + M_x\cos\phi + V\,(\varepsilon\,x - z')\cos\phi + H\,(\varepsilon\,x - z')\sin\phi]\,v \quad \ldots(34f)$$

If a unit force, P_x, is placed at the right abutment, the thrust on any block length, ds, making an angle, ϕ, with the horizontal is,

$$T = p_x\cos\phi\,dv \quad \ldots(35)$$

Table 5, which gives the forces and moments at each section due to forces and moments at the abutments, may be derived with the guidance of Fig. 15.

TABLE 5.—FORCES AND MOMENTS AT VARIOUS SECTIONS OF AN ARCH DUE TO ABUTMENT REACTIONS

(1)	x (2)	y (3)	z (4)	α (5)	β (6)	γ (7)
p_v	$p_x\cos\phi$	$\pm p_y\sin\phi$	.....	.....	.....	.....
p_u	$-p_x\sin\phi$	$\pm p_y\cos\phi$	.....	.....	.....	.....
p_z	.....	.....	p_x	.....	.....	.....
n_z	$p_x(y-h)$	$\mp p_y\,(l+x)$	.....	n_z	.....	.....
n_u	$p_x\,(\epsilon\,l+\epsilon\,x)\cos\phi$	$\pm p_y\,(\epsilon\,l+\epsilon\,x)\sin\phi$	$-p_x\,(v+l\cos\phi-h\sin\phi)$	.....	$\pm n_y\cos\phi$	$n_x\sin\phi$
n_v	$-p_x\,(\epsilon\,l+\epsilon\,x)\sin\phi$	$\pm p_y\,(\epsilon\,l+\epsilon\,x)\cos\phi$	$+p_x\,(u\pm l\sin\phi+h\cos\phi)$	.....	$\pm n_y\sin\phi$	$n_x\cos\phi$

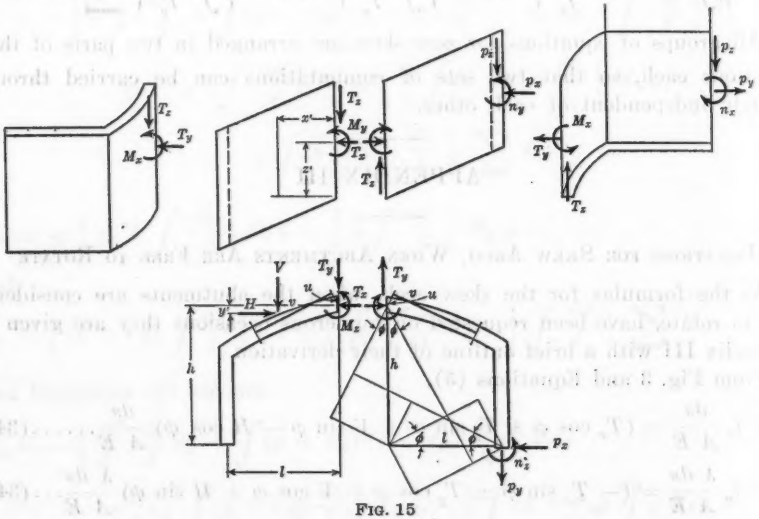


FIG. 15

By adding the work performed by the thrust, the shears (both radial and horizontal), the moments about the horizontal and radial axes, and the torque,

the total work performed in deforming this block may be determined. This is equal to the work performed by the force, p_x, moving through the distance that the abutment is deflected in the direction in which the force is acting.

If p_x equals unity, the expression for the deflection of the abutment in the direction, X, due to the deformation of one block of length, ds, is:

$$d\,\Delta = p_v\,dv + p_u\,du + p_z\,dz + n_z\,d\alpha + n_u\,d\beta + n_v\,d\gamma \ldots\ldots (36)$$

The total deflection is the integral of the several terms producing Δ.

In obtaining $d\,\Delta$ to find the expressions for p_v, p_u, n_z, etc., for a deflection in the direction, X, only those terms in Column (2) Table 5 are used; for the y-deflection, Column (3) is used, etc.

Collecting terms and using the values of Δ_x, Δ_y, etc., as defined in Equations (20), the several deflections of the right abutment of the arch when considered as fixed at the left abutment, are:

$$\Delta x - h\,\Delta\alpha + \varepsilon\,l\,\Delta\beta = \text{deflection of right abutment in } x\text{-direction} \ldots\ldots (37a)$$

$$\Delta y + l\,\Delta\alpha - \varepsilon\,l\,\Delta\gamma = \text{deflection of right abutment in } y\text{-direction} \ldots\ldots (37b)$$

$$\Delta z + h\,\Delta\gamma - l\,\Delta\beta = \text{deflection of right abutment in } z\text{-direction} \ldots\ldots (37c)$$

$$\Delta\alpha = \text{rotation of right abutment about } z\text{-axis} \ldots\ldots (37d)$$

$$\Delta\beta = \text{rotation of right abutment about } y\text{-axis} \ldots\ldots (37e)$$

$$\Delta\gamma = \text{rotation of right abutment about } x\text{-axis} \ldots\ldots (37f)$$

If the right sides of Equations (37) are all placed equal to zero, they represent the fixed-ended skew arch and the equations reduce at once to those already obtained for this condition, as they should. However, with the right abutment on a hinge, the axis of which is parallel to the z-axis, Equation (37d) cannot be placed equal to zero although Equations (37a), (37c), (37e), and (37f) may; and these four equations reduce to:

$$\Delta x - h\,\Delta\alpha = 0 \ldots\ldots (38a)$$

$$\Delta z = 0 \ldots\ldots (38c)$$

$$\Delta\beta = 0 \ldots\ldots (38e)$$

$$\Delta\gamma = 0 \ldots\ldots (38f)$$

With the left abutment hinged Equations (37b) and (37d) can be produced from purely static considerations, and a little algebraic transforming produces the following equations:

$$T_y = -\frac{1}{2\,l}(l - x')\,V - \frac{1}{2}\frac{1}{l}(h - y')\,H \ldots\ldots (39a)$$

$$Q\,M_y = -F\,T_y + e_v + e_h \ldots\ldots (39b)$$

$$A'\,T_x + C\,T_z + F\,M_x = a_v + a_h \ldots\ldots (39c)$$

$$C\,T_x + L\,T_z + O\,M_x = c_v + c_h \ldots\ldots (39d)$$

$$F\,T_x + O\,T_z + S\,M_x = f_v + f_h \ldots\ldots (39e)$$

$$M_z = \frac{1}{2}(l - x')\,V - h\,T_x + \frac{1}{2}(h - y')\,H \ldots\ldots (39f)$$

The formulas for the coefficients in Equations (39), are:

$$A' = A - 2\,h\,D + P\,h^2 = \sum \cos^2\phi\,\frac{S}{A} + \lambda \sum \sin^2\phi\,\frac{S}{A} + \sum y^2\,\frac{S}{I_z} + \varepsilon^2 \sum x^2 \cos^2\phi\,\frac{S}{I_y} + \varepsilon^2\,\lambda \sum x^2 \sin^2\frac{S}{F} - 2\,h \sum y\,\frac{S}{I_z} + h\,\varepsilon^2 \sum \frac{S}{I_z} \quad (40a)$$

$$C = -\left\{ \varepsilon \sum v\,x \cos\phi\,\frac{S}{I_y} + \varepsilon\,\lambda \sum u\,x \sin\phi\,\frac{S}{F} \right\} \quad (40b)$$

$$F = \varepsilon \sum x \sin\phi \cos\phi\,\frac{S}{I_y} - \varepsilon\,\lambda \sum x \sin\phi \cos\phi\,\frac{\lambda}{F} \quad (40c)$$

$$L = \sum v^2\,\frac{S}{I_y} + \lambda \sum u^2\,\frac{S}{F} + \lambda \sum \frac{S}{A} \quad (40d)$$

$$O = -\left\{ \sum v \sin\phi\,\frac{S}{I_y} - \lambda \sum u \cos\phi\,\frac{S}{F} \right\} \quad (40e)$$

$$Q = \sum \cos^2\phi\,\frac{S}{I_y} + \lambda \sum \sin^2\phi\,\frac{S}{F} \quad (40f)$$

and,

$$S = \sum \sin^2\phi\,\frac{S}{I_y} + \lambda \sum \cos^2\phi\,\frac{S}{F} \quad (40g)$$

In Equations (39) the terms due to vertical loads are:

$$a_v = -\frac{1}{2}\left\{ (l - x')\left[\sum_A^B y\,\frac{S}{I_z} - h \sum_A^P \frac{S}{I_z}\right] + (1-\lambda)\sum_A^P \sin\phi\cos\phi\,\frac{S}{A} - \sum_A^P x\,y\,\frac{S}{I_z} + h\sum_A^P x\,\frac{S}{I_z} - h\,x'\sum_A^P \frac{S}{I_z} + x'\sum_A^P y\,\frac{S}{I_z} + \varepsilon^2 \sum_A^P x^2 \sin\phi\cos\phi\,\frac{S}{I_y} - \varepsilon^2\,\lambda \sum_A^P x^2 \sin\phi\cos\phi\,\frac{S}{F} - \varepsilon^2\,x'\sum_A^P x \sin\phi\cos\phi\,\frac{S}{I_y} + \varepsilon^2\,x\;\lambda \sum_A^P x \sin\phi\cos\phi\,\frac{S}{F} \right\} V \quad (41a)$$

$$c_v = -\frac{1}{2}\left\{ -\varepsilon \sum_A^P v\,x \sin\phi\,\frac{S}{I_y} + \varepsilon\,\lambda \sum_A^P u\,x\cos\phi\,\frac{S}{F} + \varepsilon\,x'\sum_A^P v \sin\phi - \varepsilon\,x'\,\lambda \sum_A^P u\cos\phi\,\frac{S}{F} \right\} V \quad (41b)$$

$$f_v = -\frac{1}{2}\left\{ \varepsilon \sum_A^P x \sin^2\phi\,\frac{S}{I_y} + \varepsilon\,\lambda \sum_A^P x\cos^2\phi\,\frac{S}{F} - \varepsilon\,x'\sum_A^P \sin^2\phi\,\frac{S}{I_y} - \varepsilon\,x'\,\lambda \sum_A^P \cos^2\phi\,\frac{S}{F} \right\} V \quad (41c)$$

and,

$$e_v = -\frac{1}{2}\left\{ \varepsilon \sum_A^P x \sin\phi\cos\phi\left(\frac{S}{I_y} - \frac{\lambda S}{F}\right) - \varepsilon\,x'\sum_A^P \sin\phi\cos\phi\,\left(\frac{S}{I_y} - \frac{\lambda S}{F}\right) V \right. \quad (41d)$$

In Equations (39) the terms due to horizontal loads are:

$$a_h = -\frac{1}{2}\Big\{(h-y')\Big[\sum_A^P y\,\frac{S}{I_z} - h\sum_A^P \frac{S}{I_z}\Big] - \sum_A^P \cos^2\phi\,\frac{S}{A} - \lambda\sum_A^P \sin^2\phi\,\frac{S}{A} - \sum_A^P y^2\,\frac{S}{I_z} + y'\sum_A^P y\,\frac{S}{I_z} + h\sum_A^P y\,\frac{S}{I_z} - h\,y'\sum_A^P \frac{S}{I_z} - \varepsilon^2\sum_A^P x^2\cos^2\phi\,\frac{S}{I_y} + \varepsilon^2\,x'\sum_A^P x\cos^2\phi\,\frac{S}{I_y} - \varepsilon^2\sum_A^P x^2\sin^2\phi\,\frac{\lambda\,S}{F} + \varepsilon^2\,x'\sum_A^P x\sin^2\phi\,\frac{\lambda\,S}{F}\Big\}H \quad \ldots\ldots (42a)$$

$$c_h = -\frac{1}{2}\Big\{\varepsilon\sum_A^P v\,x\cos\phi\,\frac{S}{I_y} - \varepsilon\,x'\sum_A^P v\cos\phi\,\frac{S}{I_y} + \varepsilon\sum_A^P x\,u\sin\phi\,\frac{\lambda\,S}{F} - \varepsilon\,x'\sum_A^P u\sin\phi\,\frac{\lambda\,S}{F}\Big\}H \quad \ldots\ldots (42b)$$

$$f_h = -\frac{1}{2}\Big\{-\varepsilon\sum_A^P x\cos^2\phi\,\frac{S}{I_y} - \varepsilon\sum_A^P x\sin^2\phi\,\frac{\lambda\,S}{F} + \varepsilon\,x'\sum_A^P \cos^2\phi\,\frac{S}{I_y} + \varepsilon\,x'\sum_A^P \sin^2\phi\,\frac{\lambda\,S}{F}\Big\}H \quad \ldots\ldots (42c)$$

and,

$$e_h = -\frac{1}{2}\Big\{-\varepsilon\sum_A^P x\cos^2\phi\,\frac{S}{I_y} - \varepsilon\sum_A^P x\sin^2\phi\,\frac{\lambda\,S}{F} + \varepsilon\,x'\sum_A^P \cos^2\phi\,\frac{S}{I_y} + \varepsilon\,x'\sum_A^P \sin^2\phi\,\frac{\lambda\,S}{F}\Big\}H \quad \ldots\ldots (42d)$$

For symmetrical loading, $T_y = 0$ and $M_y = 0$. Therefore e_v and e_h need not be determined.

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PAPERS

RUN-OFF—RATIONAL RUN-OFF FORMULAS

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Synopsis

A discussion of methods of estimating storm-water run-off, statements, conclusions and comparisons, weighing of errors, advantages and disadvantages, and suggestions for improvement, form an important part of this paper.

Logically, the paper is divided into two parts. Part I contains an analysis of the factors involved; it presents conclusions, and proposes changes in the present methods of estimating run-off. Part II contains a series of derivations, introducing and advocating new factors and formulas. This part of the paper also contains practical examples and suggestions for the guidance of the designer.

PART I.—ANALYSIS OF VARIABLE FACTORS

Introduction

Many engineers are engaged in the design of culverts, bridges, storm drains, dams, and various structures, in which the first, and probably the most difficult, problem encountered is that of estimating the maximum storm, or flood, water flow to be expected from a given drainage area.

Being indeterminate, the problem is made more difficult by lack of observation and the present uncertainties in available data for the determination of values for the many varying factors that contribute to flow. Individual judgment must always play a large part in selecting values for these factors, and, consequently, some variation in results may be expected, regardless of the method used.

Nevertheless, it should be the duty of all engineers engaged in the estimation of run-off, not only to strive for the more accurate determination of values for the various factors involved, but also for better methods of applying values when determined.

The profession owes much to the engineers who evolved the various empirical formulas, which were the natural fore-runners of the more recent, so-called,

Note.—Written discussion on this paper will be closed in August, 1931, *Proceedings*.

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rational method. One purpose of this paper, however, is to show why the use of these empirical formulas is not now to be recommended.

The rational method recognizes the element of time in the computation of storm flow and, although there is much room for difference in assumptions, causing disagreement in results, it is considered by engineers to offer the most reasonable approach to accuracy. This method in some form is, therefore, rapidly supplanting the use of empirical formulas. The present users of the rational method may be placed in two main classes: First, those who follow the detail method as outlined subsequently; and second, those who attempt to arrive at a time of concentration for the water-shed either by a comparison with tests on areas of somewhat similar characteristics, or, by roughly estimating the average velocity of the main channel of flow.

The detail method is slow and tedious, and the writers believe it to be erroneous under the customary assumptions. On the other hand, in attempting to determine average velocities for the purpose of obtaining the time of concentration, and, finally, the intensity of rainfall, by either dividing a large water-shed into only a few parts, or, by comparing them with other areas of somewhat similar characteristics, results may be found to be no more accurate than those obtained by the empirical formulas.

There is no uniformity in the conception of methods for deriving the various factors involved in empirical formulas, and in assumptions made in the present use of the rational method. If a detail process had existed, which could have been used as a criterion of accuracy, the writers would have needed on much of their work, a rational formula for obtaining a reasonable result for a water-shed, without "tabling" the area in detail. Many times the engineer must know the probable maximum run-off at one or more points on a water-shed, as, for instance, in designing a bridge, culvert, or dam, or a section of drain with large tributary area. There are also many occasions for the use of such formulas for preliminary and comparative estimates when the engineer is not justified in taking the time and going to the expense of making a long, detailed calculation of the run-off. There should be a rational method of arriving at a close approximation of the run-off from a given area by a more direct use of the several factors involved. The writers, therefore, have made the following analysis, resulting in formulas which may be used for run-off computations. It is also hoped that the paper will present a clearer and more uniform conception of the rational method; and that the inadequacy of the empirical formulas for general run-off computations will be made evident.

It should be stated here that no attempt has been made to set up a "mill" into which the various factors could be thrown, a crank turned, and a perfect result obtained. The computation of run-off will always require the best judgment of experienced engineers, and the data in this paper are submitted only with the hope that their intelligent use may be a time-saver in guiding the judgment of the engineer to a rational result.

Notation

Throughout this work the writers have endeavored to concentrate on original analysis, using previously published information only when necessary for

clarity. The reader, therefore, is assumed to have a certain familiarity with existing methods for computing run-off. A complete list of the notation introduced in Part II of this paper is arranged in Appendix I. The notation for Part I occurs in the body of the paper as the analysis progresses.

Empirical Formulas

Various empirical formulas for estimating storm flow have been presented from time to time, to the Engineering Profession, and not a few of the leading cities use one of these formulas in estimating storm flow. Many engineers also use one or more of the formulas for estimating, checking, and comparing with local data, the run-off for the larger flood-control projects.

Presentation of the best known formulas by Metcalf and Eddy includes a clear account of their derivation, as well as a comparison of results obtained by their application. A wide difference in results is shown,[3] but this is not surprising when it is noted that no one formula, in the process of its derivation, includes all the varying factors contributing to the result sought.

It is not necessary to dwell at length upon these formulas, but it should be interesting to observe that, because of three considerations, the very nature of their construction precludes any possibility of allowing for the difference in flow, on areas of the same acreage and slope of the main channel of flow. These considerations are:

(1) Shape of area both as to external boundary and internal arrangement of drainage courses leading to the point of concentration;
(2) Condition and shape of the main channel of flow; and
(3) In addition to Considerations (1) and (2), the initial or inlet time cannot be allowed to vary in the empirical formulas, due to their construction.

A better conception of the effect of these three considerations, and others, is given subsequently in this paper. (See, "Inadequacy of Empirical Formulas".)

The Rational Method

The Fundamental Relation.—The relation between run-off and rainfall is, of course, correctly stated by the expression,

$$Q = C\,i\,A \qquad (1)$$

in which, Q, C, i, and A are as defined in Appendix I.

Attempts by engineers to solve this equation have evolved the so-called rational method. This method is based on the assumption that the maximum rate of run-off from any water-shed for a given rainfall intensity occurs when all parts of the area are contributing. To fulfill this condition, it follows that the rain must continue a length of time equal to, or exceeding, the time it takes water to travel from the most remote portion of the water-shed to its outlet. This is known as the time of concentration for the water-shed.

Through the study of rainfall records, certain relations have been found to exist between intensity and time. It has been generally shown that a smaller average intensity may be expected for a longer period of time than

[3] "American Sewerage Practice," Vol. I, First Edition, by Metcalf and Eddy, p. 235.

for a short one. Probably the most common form of this relation in use for estimating run-off is,

$$i = \frac{K}{t^e} \quad \ldots\ldots\ldots\

In their study of rainfall records, the writers and others have found that an individual storm may begin or end in any manner, no definite general tendency as regards the order of greatest intensity having been observed.

The rainfall curves generally in use (including the construction of the various expected frequency periods, as 10, 15, 25, etc., years), are built on the basis of an average intensity for a given period, such intensity being determined from records of storms of various types as regards the relative intensity within a given time period. It is clear, therefore, that because a curve may have the form of Equation (2), there is no reason to assume that an individual storm will follow the law of this curve. In using curves so constructed the type of storm for a given time of concentration should be considered as independent of the law of the curves. To construct curves, the use of which is to imply that the type of storm to be expected for a given time of concentration is one whose variation in intensity for that time follows a definite law, would require the selection of data for the various time intervals from storms the rates of precipitation of which actually followed the particular law. The probable frequency periods would depend necessarily on the type of storm selected. This would be impracticable; and it would seem that the method now followed in making curves offers a rational attack, since the type of storm actually used approaches one of uniform intensity for the time of concentration.

This conception should be concurred in by those who pursue the detail process of the rational method in storm drain estimates, in which it is the evident intention to assume that the intensity is uniform over the entire watershed throughout the time of concentration. Such assumption, no doubt, would conform to that of C. E. Ramser, M. Am. Soc. C. E., of the U. S. Bureau of Public Roads, who states;[4]

"The maximum rate of run-off, therefore, would result from a rainfall of maximum uniform intensity, continuing for a time equal to or exceeding the time of concentration."

The assumption of a uniform, or constant, intensity for a given time of concentration, affording the most feasible means of building rainfall curves to forecast the probable intensities and frequencies of storms, does not, of course, preclude the possible occurrence of precipitation varying in intensity within a given period of time; but a storm, following any definite law (in regard to the relative intensity within the time period) other than that of the average storm used in constructing the rainfall curves, may be expected to have a different frequency of occurrence. Hence, the 10-year, 15-year, 25-year, etc., frequency periods of the curves, as ordinarily constructed, cannot properly be used for forecasting storms of varying types.

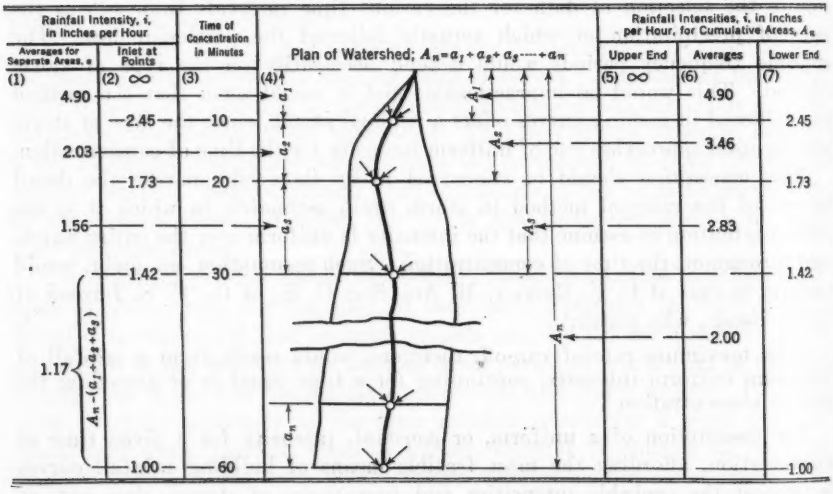
The Detail Method.—In solving by the detail method, it is first necessary to determine the time it takes the water to flow from the most remote portion of the first small area, or nucleus, to its point of inlet to the main channel of flow. For this time the intensity, i, is determined from the rainfall curve in use, the value, Q, is calculated from Equation (1) and its time of flow to the next inlet point is determined. The time of concentration for the total area

[4] "Run-Off from Small Agricultural Areas," reprinted from *Journal of Agricultural Research*, Vol. 34, No. 9, p. 799.

contributing at or above this second inlet point equals the concentration time for the first area plus the time of flow in the drain to the second inlet point. The rate of run-off, Q, is then calculated for the second inlet point, using the new time which gives a decreased intensity for the entire area contributing at, or above, the second inlet point, etc., at all inlet points until the entire water-shed is traversed.

Merely for the purpose of illustrating the use of the rainfall curve as generally practiced in the detail method, a simple water-shed is chosen. In Table 1 the points enclosed with circles represent inlet points on the main drainage channel, which is designated by the heavy line.

TABLE 1.—COMPUTATION TABLE FOR STUDY OF RAINFALL INTENSITY BY MEANS OF RAINFALL CURVE



Let the total areas contributing water above the successive inlet points equal A_1, A_2, A_3, etc.; times of concentration, t_1, t_2, t_3, etc.; and corresponding average intensities, i_1, i_2, i_3, etc., respectively.

Let the individual areas contributing water to the successive inlet points be represented by a_1, a_2, a_3, etc. (shown on left of Table 1). Then, let i'_1, i'_2, i'_3, etc., equal the average intensity of rainfall on the areas a_1, a_2, a_3, etc., at the respective times, t'_1, t'_2, t'_3, etc. For this simple water-shed it is assumed that the time of concentration of the individual areas is approximately equal to the differences in time between inlet points on the main line, or, that the intensities, i'_1, i'_2, i'_3, etc., are due to the time periods, 0 to t_1, t_1 to t_2, t_2 to t_3, etc., respectively.

Considering Area A_n, Table 1, and following strictly the assumption of the rational method, that the maximum flow occurs when the storm lasts as long, or longer, than the time required for the water to travel from the most remote

portion of the water-shed to its outlet, enabling all parts of the area to contribute simultaneously, then, the run-off from Area a_1, produced by Intensity i'_1, occurring at Time t'_1 will reach the outlet when Area a_n is contributing, due to Intensity i'_n falling at Time t'_n.

Likewise, the run-offs from Areas a_2, a_3, etc., produced by Intensities i'_2, i'_3, etc., occurring at Times t'_2, t'_3, etc., respectively, will reach the outlet when Area a_n is contributing, due to Intensity i'_n falling at Time t'_n. The total discharge at the outlet of Area A_n will equal the sum of the discharges from the individual areas composing the water-shed.

Now, solely for the purpose of illustrating the present use of the rainfall curve in the detail method, it will be assumed that the run-off coefficient is constant throughout, say, equal to one. Any form of Equation (2) may be selected. For example, let $e = 0.5$ and (when t is expressed in seconds) let $K = 120$. Values of average rainfall intensities, corresponding to successive abscissas of time, may be determined and plotted as ordinates in Fig. 1.

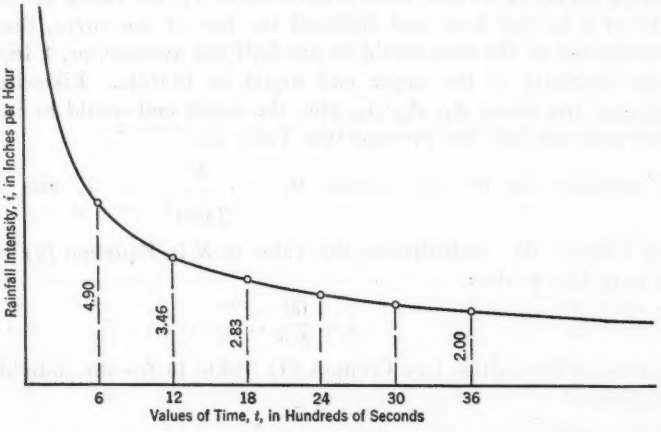


FIG. 1.—ILLUSTRATIVE CURVE OF RAINFALL INTENSITIES, $i = \frac{120}{t^{0.5}\text{ sec.}}$

Under the foregoing assumptions, the water-shed, A_n, is expected to receive an average intensity of 2 in. per hour for the time of 60 min. If this general detail method is followed, the time, 60 min., is determined by velocities, produced by the use of the intensities, 4.90, 3.46, 2.83, ... 2.0, for the areas, A_1, A_2, A_3, ... A_n, respectively.

Now, if the storm is assumed to have a uniform intensity for the period of time, t_n then, $i_1 = i_2 = i_3$, ... $= i_n$, and the velocities that produce the time, t_n, in the computations, will be the result of a uniform, or constant, intensity for the time of the water-shed.

Thus, although it is the evident intention to assume a uniform intensity over the entire area throughout the time of concentration, the process followed in the detail method automatically assumes the equivalent of a variable intensity during the period.

The particular type of storm thus automatically assumed, not only depends on the physical characteristics, and the shape and manner of concentration

of the water-shed, but also on the type of rainfall curve in use. When the area is shaped so that the time of concentration increases proportionately with the contributing area, the type of storm automatically assumed follows the law of the curve in use.

Assume, for the purpose of illustration, a rainfall curve of the type,

$$i = \frac{K}{t^{0.5}} \quad \ldots\ldots (3)$$

The average value of the ordinate, i, may be found as follows: When $y = f(x)$, or $i = f(t)$, the expression for the area bounded by the curve, the outer ordinate, and the co-ordinate lines, is $\int f(x)\,dx = \int k\,t^{-0.5}\,dt = \frac{K\,t^{0.5}}{0.5}$ $= 2\,K\,t^{0.5}$. Then, $2\,K\,t^{0.5}$ divided by the abscissa, t, $= \frac{2\,K\,t^{0.5}}{t} = \frac{2\,K}{t^{0.5}}$. Therefore, the average ordinate equals twice the end ordinate.

If, for a period of 60 min. on the water-shed, A_n, the storm had an average intensity of 2 in. per hour and followed the law of the curve, the intensity at the lower end of the area would be one-half the average, or, 1 in. per hour, while the intensity at the upper end would be infinite. Likewise, for the intensities of the areas, A_1, A_2, A_3, etc., the upper end would be infinite and the lower end, one-half the average (see Table 1).

The intensity for 60 min. equals $R_1 = \frac{K'}{3\,600^{\frac{1}{2}}} = 1$, and therefore, $K' = \sqrt{3\,600} = 60$. Substituting this value of K in Equation (2) and letting $e = 0.5$ as in Fig. 1; then,

$$i = \frac{60}{t^{0.5}} \quad \ldots\ldots (4)$$

The average intensities (see Column (1) Table 1) for the individual areas are as follows:

For a_1,

$$\frac{2\,K'}{\sqrt{t_1}} = 4.90$$

For a_2,

$$\frac{\left(2\,K'\,t_2^{\frac{1}{2}} - 2\,K'\,t_1^{\frac{1}{2}}\right)}{(t_2 - t_1)} = 2.03$$

For a_3,

$$\frac{\left(2\,K'\,t_3^{\frac{1}{2}} - 2\,K'\,t_2^{\frac{1}{2}}\right)}{(t_3 - t_2)} = 1.56$$

and, for $A_n - (a_1 + a_2 + a_3)$,

$$\frac{\left(2\,K'\,t_n^{\frac{1}{2}} - 2\,K'\,t_3^{\frac{1}{2}}\right)}{(t_n - t_3)} = 1.17$$

Assume any values for the individual areas in Table 1 that will vary directly as the time, such as $a_1 = 100$, $a_2 = 100$, $a_3 = 100$, and $A_n - (a_1 - a_2 - a_3) = 300$ acres.

In Equation (1) let $C = 1$; then, the "Q-collecting" at the outlet of the water-shed would be as shown in Columns (2) and (3) of Table 2, with proper values of i and a substituted for each successive case. The average intensity would be $\frac{1\,200}{600} = 2$ in. per hour, which is the value used in the detail method. However, this process will produce the same time of concentration as the detail method, when and only when the area varies directly with the time.

When the area increases at a faster rate than time, the results are as illustrated in Columns (4) and (5) of Table 2. In this case, 1 900 acres $\times$ 2.00 in. per hour = 3 800 cu. ft. per sec., which is greater than the rate of run-off obtained by assuming that the storm obeyed the law expressed by the curve in use. Similarly, (Columns (6) and (7)), 200 acres $\times$ 2.00 in. per hour = 400 cu. ft. per sec., which is less than the rate of run-off obtained by assuming that the storm obeyed the law expressed by Equation (2).

TABLE 2.—RATES OF RUN-OFF, Q, FOR THREE TYPES OF WATER-SHED

Intensity of rainfall, in inches per hours	AREA, a, AND TIME, t, INCREASING PROPORTIONATELY		AREA, a, INCREASING FASTER THAN TIME, t		TIME, t, INCREASING FASTER THAN AREA, a	
	a, in acres	Q, in cubic feet per second	a	Q	a	Q
(1)	(2)	(3)	(4)	(5)	(6)	(7)
4.90	100	490	100	490	100	490
2.03	100	203	300	609	50	101.5
1.56	100	156	500	780	30	46.8
1.17	300	351	1 000	1 170	20	23.4
Total.....	600	1 200	1 900	3 049	200	661.7

Conclusions.—The foregoing observations lead to four major conclusions as follows:

1.—For a constant value of the run-off coefficient, and for the special type of water-shed assumed in Table 1, three facts are noted:

(*a*) When the time of concentration increases proportionately with the contributing area, the type of storm assumed follows the law of the rainfall curve in use.

(*b*) When the area increases at a faster rate than the time, the run-off obtained by the detail method is greater than that obtained when the type of storm is assumed to follow the law of the curve in use.

(*c*) When the area increases at a slower rate than the time of concentration, the run-off, Q, obtained by the detail method is less than that obtained when the type of storm is assumed to follow the law of the curve in use.

2.—In all cases, the detail method automatically assumes the equivalent of a storm of varying intensity, causing a smaller time of concentration because of a greater average velocity of flow (and, consequently, a greater value of Q) than would be obtained by assuming a storm of uniform (or constant) intensity for the time of the water-shed. The use of the rainfall curve is, therefore, inconsistent with its making, in that while the curve represents a certain period of expected frequency based on types of storms occurring in any variety of ways (averaging in effect a uniform intensity for a given period of time), it is really used to accommodate a system of tabling which automatically assumes a different type of storm for every type of rainfall curve and water-shed.

3.—Consider a storm that varies in intensity within the time of concentration, but is evenly distributed over the entire area at any moment. If the run-off coefficient is constant throughout the water-shed, the summation of discharges from the individual areas is the same as that due to a rainfall with a uniform rate of precipitation (equal to the average of the preceding rates) over the water-shed for the time period, when (and only when) the contributing area increases proportionately with the time.

4.—It is apparent, therefore, that neither the assumption of a uniform intensity over the water-shed for the time of concentration, nor its approach through the medium of the detail method, affords, in all cases, the absolute maximum run-off.

Statement.—It is manifestly the intention of those who use the detail rational method to assume that the expected storm is one of uniform intensity over the entire area throughout the time of concentration. The afore-mentioned inconsistency, affording the variable factor of safety, is due to the use of velocities that do not obtain under the assumption. Practically, however, this factor of safety is not of major importance, being usually only a small percentage. Practical visualization of its probable magnitude may be had from a comparison of the values of the factor, P, in Table 3 with those in Table 4, raised to the exponent indicated in Equations (28) and (29), given subsequently. The values shown in Table 3 are based on the assumption that the type of storm is one of uniform intensity over the area throughout the time of concentration; while those in Table 4 resulted from the use of the rainfall curve, of the type, $i = \frac{K}{t^{0.5}}$, as ordinarily practiced in the detail rational method. It should be mentioned that this factor of safety is varied somewhat by the type of rainfall curve; it may be somewhat greater for a given water-shed, for instance, when e in Equation (2) is 0.7 than when e is 0.5 as in Equation (3).

In order to arrive at a greater run-off than that rendered by the assumption of a uniform intensity for the area throughout the time of concentration, it would be necessary to assume a different manner of variation in intensity, within the period, for every water-shed. The peak, or rather the peaks, of the storm, for the period of concentration, would then be so arranged that the contributions from the individual areas composing the water-shed might

render the greater run-off. Intensities are relative, however, and there is no limiting peak, either in height or duration. There is, then, no absolute maximum for a given water-shed; and, although it is possible to arrive at a greater run-off than that to be expected from the rational assumption, the impracticability of predicting the probable frequency of occurrence of storms particularly selected for every type of water-shed, the difficulty in determining the effect on the run-off coefficient, and the variability of the time of concentration, due to all physical characteristics of a water-shed and changes in intensity, complicate the problem to a measure almost beyond practical solution.

TABLE 3.—VALUES OF THE FACTOR, P, FOR STORMS OF UNIFORM INTENSITY

$\frac{A}{A'}$	$\frac{L}{W}$	$P' = 2.0$			$P' = 1.0$			$P' = 0.5$			$P' = 0.3$			$P' = 0.1$		
		Concentration			Concentration			Concentration			Concentration			Concentration		
		Delayed	Medium	Advanced	Delayed	Medium	Advanced	Delayed	Medium	Advanced	Delayed	Medium	Advanced	Delayed	Medium	Advanced
16	2	0.76	0.79	0.87	0.59	0.64	0.69	0.44	0.46	0.48	0.30	0.31	0.32	0.11	0.11	0.12
	3	0.75	0.80	0.88	0.60	0.65	0.70	0.46	0.48	0.51	0.34	0.35	0.37	0.13	0.14	0.14
	4	0.74	0.81	0.89	0.61	0.66	0.71	0.48	0.50	0.53	0.36	0.37	0.39	0.14	0.15	0.15
	8	0.72	0.78	0.89	0.64	0.67	0.72	0.53	0.55	0.56	0.42	0.44	0.46	0.20	0.21	0.21
	16	0.72	0.76	0.79	0.65	0.68	0.73	0.56	0.57	0.58	0.48	0.49	0.50	0.27	0.28	0.28
64	2	0.62	0.67	0.72	0.55	0.58	0.63	0.43	0.45	0.48	0.33	0.35	0.37	0.16	0.16	0.17
	3	0.63	0.67	0.73	0.56	0.59	0.64	0.45	0.48	0.50	0.36	0.38	0.40	0.18	0.19	0.19
	4	0.63	0.68	0.73	0.57	0.60	0.65	0.47	0.50	0.52	0.38	0.40	0.42	0.20	0.21	0.22
	8	0.64	0.69	0.74	0.60	0.63	0.66	0.52	0.54	0.57	0.44	0.46	0.48	0.25	0.26	0.27
	16	0.66	0.69	0.72	0.62	0.65	0 67	0.56	0.58	0.60	0.50	0.51	0.52	0.31	0.32	0.32
256	2	0.55	0.58	0.63	0.50	0.52	0.57	0.42	0.44	0.47	0.35	0.37	0.38	0.18	0.20	0.20
	3	0.55	0.59	0.64	0.51	0.54	0.58	0.45	0.47	0.50	0.38	0.40	0.42	0.22	0.22	0.23
	4	0.57	0.60	0.65	0.52	0.55	0.59	0.47	0.50	0.52	0.40	0.42	0.44	0.24	0.25	0.26
	8	0.58	0.63	0.67	0.56	0.60	0.61	0.51	0.54	0.57	0.46	0.48	0.49	0.29	0.30	0.31
	16	0.62	0.65	0.68	0.60	0.61	0.62	0.56	0.58	0.61	0.51	0.52	0.53	0.35	0.36	0.37
4 096	2	0.48	0.51	0.55	0.47	0.47	0.53	0.42	0.44	0.47	0.36	0.40	0.41	0.24	0.27	0.27
	3	0.49	0.52	0.56	0.48	0.50	0.54	0.44	0.47	0.50	0.41	0.43	0.44	0.29	0.29	0.32
	4	0.51	0.53	0.57	0.49	0.51	0.55	0.46	0.49	0.52	0.43	0.46	0.46	0.32	0.32	0.35
	8	0.54	0.58	0.62	0.53	0.56	0.58	0.51	0.54	0.57	0.49	0.51	0.53	0.38	0.39	0.41
	16	0.59	0.62	0.64	0.58	0.58	0.60	0.56	0.59	0.62	0.53	0.54	0.56	0.42	0.43	0.44
262 144	2	0.45	0.47	0.50	0.45	0.45	0.50	0.42	0.44	0.47	0.39	0.43	0.44	0.32	0.34	0.34
	3	0.46	0.48	0.51	0.46	0.48	0.51	0.44	0.47	0.49	0.42	0.45	0.48	0.37	0.38	0.40
	4	0.48	0.49	0.52	0.47	0.49	0.52	0.46	0.49	0.51	0.44	0.48	0.50	0.40	0.42	0.44
	8	0.52	0.55	0.57	0.51	0.54	0.56	0.50	0.53	0.57	0.50	0.53	0.55	0.47	0.48	0.49
	16	0.57	0.60	0.62	0.56	0.57	0.58	0.56	0.59	0.62	0.54	0.55	0.57	0.50	0.51	0.52

Assuming a uniform intensity for the entire area throughout the time of concentration enables the engineer to forecast with reasonable accuracy the probable frequency of this average storm. The solution of the run-off problem by means of rainfall curves, showing the relation between time and intensity for various periods of expected frequency, therefore, may, be adopted as the standard basis for economical design. It may reasonably be considered that the possible arrangement of intensities, within the time of concentration, to render a greater run-off, constitutes a very rare occurrence, which (in special cases only) may be taken care of by a special factor of safety.

TABLE 4.—VALUES OF THE FACTOR, P, FROM USE OF RAINFALL CURVE OF THE TYPE, $i = \frac{K}{t^{0.5}}$

$\frac{A}{A'}$	$\frac{L}{W}$	$P' = 2.0$			$P' = 0.9$			$P' = 0.5$			$P' = 0.3$			$P' = 0.1$		
		Concentration			Concentration			Concentration			Concentration			Concentration		
		Delayed	Medium	Advanced	Delayed	Medium	Advanced	Delayed	Medium	Advanced	Delayed	Medium	Advanced	Delayed	Medium	Advanced
16	2	0.88	0.92	0.99	0.63	0.67	0.72	0.46	0.49	0.52	0.32	0.35	0.38	0.12	0.13	0.14
	3	0.86	0.92	0.99	0.65	0.70	0.74	0.49	0.52	0.55	0.35	0.38	0.41	0.15	0.15	0.16
	4	0.85	0.91	0.99	0.67	0.71	0.75	0.52	0.54	0.57	0.37	0.40	0.43	0.17	0.17	0.18
	8	0.83	0.90	0.96	0.70	0.74	0.78	0.57	0.60	0.63	0.44	0.47	0.50	0.24	0.24	0.25
	16	0.82	0.89	0.94	0.73	0.76	0.80	0.62	0.65	0.68	0.51	0.54	0.56	0.31	0.32	0.33
64	2	0.72	0.76	0.82	0.58	0.62	0.66	0.46	0.49	0.52	0.34	0.37	0.40	0.16	0.17	0.18
	3	0.73	0.77	0.83	0.60	0.65	0.69	0.49	0.52	0.55	0.39	0.41	0.43	0.19	0.19	0.20
	4	0.73	0.77	0.83	0.62	0.66	0.70	0.52	0.55	0.57	0.41	0.44	0.46	0.21	0.21	0.22
	8	0.75	0.79	0.84	0.66	0.70	0.74	0.58	0.61	0.63	0.47	0.50	0.52	0.27	0.28	0.28
	16	0.76	0.81	0.85	0.70	0.73	0.77	0.63	0.66	0.69	0.52	0.56	0.58	0.33	0.34	0.35
256	2	0.62	0.66	0.72	0.55	0.58	0.62	0.46	0.49	0.52	0.37	0.40	0.43	0.21	0.22	0.23
	3	0.63	0.67	0.73	0.57	0.60	0.64	0.50	0.52	0.55	0.41	0.44	0.46	0.23	0.25	0.26
	4	0.64	0.68	0.73	0.58	0.62	0.66	0.52	0.54	0.57	0.43	0.46	0.48	0.25	0.27	0.29
	8	0.68	0.72	0.76	0.64	0.68	0.72	0.58	0.61	0.63	0.49	0.52	0.55	0.32	0.33	0.34
	16	0.72	0.76	0.79	0.68	0.72	0.75	0.63	0.66	0.69	0.55	0.58	0.61	0.38	0.40	0.41
4 096	2	0.51	0.59	0.63	0.52	0.55	0.58	0.47	0.49	0.52	0.41	0.44	0.47	0.28	0.29	0.30
	3	0.56	0.60	0.64	0.54	0.57	0.60	0.50	0.52	0.55	0.44	0.47	0.50	0.32	0.32	0.34
	4	0.58	0.62	0.64	0.55	0.58	0.62	0.53	0.55	0.57	0.47	0.50	0.52	0.34	0.35	0.37
	8	0.63	0.67	0.71	0.61	0.65	0.69	0.58	0.61	0.63	0.52	0.56	0.59	0.39	0.41	0.43
	16	0.68	0.72	0.76	0.66	0.70	0.73	0.63	0.66	0.69	0.58	0.61	0.64	0.44	0.47	0.50
262 144	2	0.50	0.54	0.58	0.50	0.53	0.56	0.47	0.49	0.52	0.44	0.47	0.51	0.36	0.38	0.41
	3	0.52	0.55	0.59	0.52	0.55	0.58	0.50	0.53	0.56	0.47	0.50	0.54	0.41	0.43	0.46
	4	0.54	0.57	0.59	0.53	0.56	0.60	0.53	0.55	0.58	0.50	0.53	0.56	0.43	0.46	0.49
	8	0.60	0.63	0.66	0.59	0.63	0.67	0.58	0.61	0.64	0.55	0.59	0.62	0.48	0.51	0.54
	16	0.65	0.69	0.72	0.64	0.68	0.71	0.64	0.67	0.70	0.61	0.64	0.67	0.52	0.57	0.61

Factor C, The Run-Off Coefficient.—Among the afore-mentioned uncertainties in available data for the estimation of storm flow, the run-off coefficient, Factor C, probably offers the widest range of values for water-sheds with similar characteristics calculated to affect this factor. Much has been written concerning this factor, and it is not here considered necessary to review all the many characteristics of a water-shed that determine its value. The writers, however, do wish to stress one main difference in assumptions for the variation in the value of the run-off coefficient as made by those who use the rational method; that is, C is assumed to vary in one of the following three ways (the type of territory as regards physical characteristics, irregularities of surface, etc., being the same):

(1) C increases as the time increases.
(2) C remains constant as the time increases.
(3) C decreases as the time increases.

Although many examples could be shown, it should suffice, for the purpose of comparison, to give one typical example of each of these assumptions.

Assumption (1) is exemplified by Table 5 which was used in design by the City of Hamilton, Ont., Canada.

TABLE 5.—RUN-OFF COEFFICIENTS USED FOR DESIGN PURPOSES IN HAMILTON, ONTARIO, CANADA

Duration of rainfall, in minutes	RUN-OFF COEFFICIENTS		
	Pervious surface	Outlying district	Central business
5	0.00	0.50	0.75
10	0.05	0.60	0.80
15	0.10	0.75	0.85
20	0.15	0.80	0.90
25	0.20	0.825	0.95
30	0.25	0.85	0.95
35	0.30	0.865	0.95
40	0.35	0.883	0.95
45	0.40	0.90	0.95
50	0.45	0.917	0.95
55	0.50	0.93	0.95
60	0.55	0.95	1.00

Table 6 used by the City of Washington, D. C., may serve as an example of Assumption (2).

TABLE 6.—RUN-OFF COEFFICIENTS USED FOR DESIGN PURPOSES IN WASHINGTON, D. C.

Characteristics	Constant run-off coefficient
Extreme suburban areas with 20 to 40% parking and widely detached houses	0.35
Suburban areas with widely detached houses	0.45
Suburban areas with widely detached houses	0.55
Areas with 50% attached and 50% detached houses	0.65
Areas closely built up	0.75
Business areas	0.85

An example to illustrate Assumption (3) has been given[5] by C. E. Grunsky, Past-President, Am. Soc. C. E., and need not be reproduced for present purposes. It should be evident to the reader, without further elaboration, that these assumptions for the value of the run-off coefficient for use in the fundamental relation, $Q = C\,i\,A$ (or its equivalent in other letter designations), may lead to widely different run-off quantities for a given water-shed, no matter how carefully and how accurately the calculations may be carried out for a given case.

On account of the many varying conditions and characteristics of water-sheds, accurate measurement of the run-off coefficient is very difficult to accomplish. Much of the available experimental information is of little value, and, no doubt, improper analysis of some of the more valuable results is often made. The writers at this point wish to urge a more unified effort on the part of the profession not only toward a thorough analysis of present available data, but also future experimentation along lines following a preconceived plan.

[5] *Transactions*, Am. Soc. C. E., Vol. LXV (1909), p. 325.

Pursuant to this thought, the writers offer the following analysis with the hope that the conclusions drawn will be helpful in future consideration of Factor *C*.

Among those who have contributed to the information on run-off, probably no one man has done more and better work than the late Emil Kuichling, M. Am. Soc. C. E. Many of those who base their computations on Assumption (1), probably have been guided by the following five conclusions drawn by Mr. Kuichling[6]:

5.—"The percentage of the rainfall discharged from any given drainage area is nearly constant for all rains of considerable intensity, and lasting equal periods of time * * *.

6.—"The said percentage varies directly with the urban development of the district; or, in other words, with the amount of impervious surface thereon * * *.

7.—"The said percentage increases rapidly, and directly or uniformly, with the duration of the maximum intensity of the rainfall, until a period is reached which is equal to the time required for the concentration of the drainage waters from the entire tributary area at the point of observation; but if the rainfall continues at the same intensity for a longer period, the said percentage will continue to increase for the additional interval of time at a much smaller rate than previously. This circumstance is manifestly attributable to the fact that the permeable surface is gradually becoming saturated and is beginning to shed some of the water falling upon it; or, in other words, the proportion of impervious surface slowly increases with the duration of the rainfall.

8.—"The said percentage becomes larger when a moderate rain has immediately preceded a heavy shower, thereby partially saturating the permeable territory and correspondingly increasing the extent of impervious surface.

9.—"The sewer discharge varies promptly with all appreciable fluctuations in the intensity of the rainfall, * * *."

There can be no doubt as to the correctness of Conclusions (6), (7), (8), and (9), but it is probable that some users of the rational method have misconstrued Conclusion (5) and perhaps Conclusion (7), or taken it too literally, in forming conclusions regarding the variation of the run-off coefficient.

For any water-shed not completely saturated from the beginning, and having constant physical characteristics throughout which are calculated to affect the run-off coefficient, the following conclusions will hold:

10.—Because of an increasing wetness (causing a higher percentage of imperviousness) the run-off coefficient for the successive individual areas composing the water-shed, increases with the time.

11.—For the intensity decreasing with the time, the run-off coefficient for the successive individual areas composing the water-shed, will decrease with the time.

12.—For all rainfalls of considerable uniform intensity on any given water-shed (the previous degree of saturation and physical characteristics remaining constant for the separate precipitations), the run-off coefficient for a rain of given intensity will be higher than that for one of less intensity, and conversely.

Referring again to Table 1, assume a simple water-shed with constant physical characteristics throughout. Let the individual areas contributing

[6] *Transactions*, Am. Soc. C. E., Vol. XX (1889), p. 37.

water to the successive inlet points be represented by a_1, a_2, a_3, etc. (see Column (1), Table 1). Let i'_1, i'_2, i'_3, etc., equal the average intensities falling on the areas, a_1, a_2, a_3, etc., at the respective times, t'_1, t'_2, t'_3, etc. Let C'_1, C'_2, C'_3, etc., represent the respective run-off coefficients.

Let the total areas contributing water above the successive inlet points equal A_1, A_2, A_3, etc. Let C_1, C_2, C_3, etc., equal run-off coefficients; t_1, t_2, t_3, etc., equal successive times of concentration; and i_1, i_2, i_3, etc., equal corresponding average intensities for A_1, A_2, A_3, etc., respectively.

Let r = rate of retention, in cubic feet per second per acre. It should be understood that retention represents that proportion of the falling water which does not run off because of evaporation, absorption, and storage, or,

$$r = i - C\,i = i\,(1 - C) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Hence, let the average rates of retention for the areas: a_1, a_2, a_3, ... $a_n = r'_1$, r'_2, r'_3, ... r'_n, respectively; and for A_1, A_2, A_3, ... $A_n = r_1$, r_2, r_3, ... r_n, respectively.

Conclusion (10) may be best illustrated by assuming a constant intensity, i, for any water-shed, as A_n. Then $(ia_1 - r'_1a_1)$, $(ia_2 - r'_2a_2)$, $(ia_3 - r'_3a_3)$, etc., represent run-off quantities, in cubic feet per second, for the areas, a_1, a_2, a_3, etc. From Equation (1), $Q = C\,i\,a$, or,

$$Q'_1 = C'_1\,i\,a_1;\ Q'_2 = C'_2\,i\,a_2;\ Q'_3 = C'_3\,i\,a_3;\ \ldots Q'_n = C'_n\,i\,a_n \ldots . (6)$$

Therefore, equating, $C'_1\,i\,a_1 = i\,a_1 - r'_1\,a_1 \ldots C'_n\,i\,a_n = i\,a_n - r'_n\,a_n$, or,

$$C'_1 = 1 - \frac{r'_1}{i} \ldots C'_n = 1 - \frac{r'_n}{i} \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Likewise, it may be inferred that, considering the areas, A_1, A_2, A_3, ... A_n,

$$C_1 = 1 - \frac{r_1}{i} \ldots C_n = 1 - \frac{r_n}{i} \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Consider the water-shed, A_n, and follow the assumption of the rational method that the maximum flow occurs when the storm lasts as long, or longer, than the time required for the water to travel from the most remote part of the water-shed to its outlet. This would enable all parts of the water-shed to contribute simultaneously; that is, run-off from Area a_1, produced by Intensity i, occurring at Time t'_1, will reach the outlet when Area a_n is contributing, due to Intensity i, falling at Time t'_n.

Likewise, the run-offs from Area a_2, a_3, etc., occurring at Times t'_2, t'_3, etc., respectively, will reach the outlet when Area a_n is contributing, due to Intensity i, falling at Time t'_n.

The rainfall that produced the run-offs from a_1, a_2, a_3, etc., lasted for the time, t'_n; but at the time run-off occurred the rain had lasted for a time, t'_1, on Area a_1, t'_2 on Area a_2, t'_3 on Area a_3, etc.

But, $t'_1 < t'_2 < t'_3, \ldots < t'_n$, and unless the water-shed was completely saturated before the rain occurred, $r'_1 > r'_2 > r'_3 \ldots > r'_n$. Therefore, $C'_1 < C'_2 < C'_3, \ldots < C'_n$, and the same process of reasoning would show that $C_1 < C_2 < C_3 \ldots < C_n$. It should be conceded, therefore, that Conclusion (10) is correct.

Conclusion (11) as applied to the rational method may be best illustrated by assuming a constant rate of retention for the individual areas composing any water-shed, such as A_n.

Then, the logic in the case of Conclusion (10) may be applied to Conclusion (11), substituting i' for i and r for r'. The result will be similar to Equation (8),

$$C_1 = 1 - \frac{r}{i_1}; \quad \ldots \quad C_n = 1 - \frac{r}{i_n} \qquad (9)$$

By inspection, it may be seen that, when $i'_1 > i'_2 > i'_3 \ldots > i'_n$, then $C'_1 > C'_2 > C'_3 \ldots > C'_n$.

For a practical visualization of the detail method the rainfall curve in Fig. 1 may be used with values of time and average intensities for Areas A_1, A_2, A_3, etc., as shown in Column (7) of Table 1. Assuming a constant value of r, say, 1.00 cu. ft. per sec. per acre and substituting in Equation (9), the values of intensities given in Table 1 (Column (7)): $C_1 = 0.80$; $C_2 = 0.71$; $C_3 = 0.65$; $\ldots$ $C_n = 0.50$.

By a somewhat similar process of reasoning, it may be shown that when r varies directly as i a constant value of C results.

Referring again to the foregoing Conclusion (5), (by Mr. Kuichling), five assumptions may be made for illustrative purposes as follows:

(4) A given drainage area;
(5) A rain of considerable intensity, say 3 in. per hour;
(6) A rain of considerable intensity, say 6 in. per hour;
(7) Equal time periods; and,
(8) A constant value of C, say, 0.50.

Then, in Equation (9), $0.50 = 1 - \frac{r}{3.00}$, and $r = 1.50$ cu. ft. per sec. per acre. When $i = 6.00$ and $C = 0.50$, $r = 3.00$ cu. ft. per sec. per acre.

Stating the foregoing Conclusion (5) by Mr. Kuichling in words would be the equivalent of saying that for a given drainage area the capacity for retention, in cubic feet per second per acre, for a rain of 6 in. per hour is double that for a rain of 3 in. per hour; or, generally, that the capacity for retention of a given area varies directly as the intensity.

It is customary in the rational method (and properly so) to assume a certain existing wetness or degree of saturation, of a given area due to previous rains. This existing degree of saturation will always be one of the main determining factors in considering this rate of retention, and, consequently, the value of Factor C, including its variation with the time. Although the requirements of any particular problem must always tax the judgment of the individual engineer, it may be possible, in a future study of the run-off coefficient (through a more unified effort of the profession), to establish some kind of standard scale of previous degree of saturation to be used as a guide to this judgment.

Reverting to the consideration of Conclusions (11) and (12), the rate of retention (assuming a certain degree of previous wetness), may be subdivided into three main component parts, namely, evaporation, absorption, and storage.

There is no reason to assume any material difference in evaporation, in cubic feet per second per acre, from a given area due to two separate rains of considerable but different intensities. Likewise, there is no reason to assume a materially greater rate of absorption, expressed in cubic feet per second per acre, on a given area with a certain previous saturation from a rain of any considerable intensity than from one of less intensity severe enough to produce general run-off.

Considering storage, expressed in cubic feet per second per acre, a given area may be assumed to have a certain degree of previous saturation. For example, assume that on such an area a rain of considerable intensity, say, 3 in. per hour occurred. It is reasonable to suppose that, in addition to evaporation and absorption, a certain quantity of water is lost to run-off for the time of concentration by storage in irregularities of surface, vegetation (if any), etc. Before producing general run-off the water must overcome these factors. If, at another time, a rain of greater intensity, say, 6 in. per hour, should fall upon the same area with the same degree of previous wetness, it would have like storage irregularities, etc. to overcome before producing general run-off.

It is, therefore, not reasonable to assume that the storage capacity of a given area (in cubic feet per second per acre), which is in condition to shed its water, is twice as great for a rain of 6 in. per hour as that for one of 3 in. per hour; or, generally, that the capacity for storage of a given area varies directly as the intensity.

The foregoing analysis points to the correctness of Conclusions (11) and (12). Experimental proof showing the correctness of Conclusions (11) and (12) is not entirely lacking[7], although much work is to be done before comprehensive values of the run-off coefficient due to changing intensities, can be established for use in the rational method.

The range in intensities encountered in run-off studies is considerable, and the present tendency of many users of the detail rational method to increase Factor C as the time increases (as shown in current publications and textbooks), without also allowing for the fluctuation in C for changing intensities, seems entirely unjustified.

Conclusions.—Based on the foregoing analysis the writers make the following conclusions, which (it is hoped) may be of some use as guides to a proper selection of Factor C (the characteristics calculated to produce run-off and the previous degree of saturation are assumed to be constant for the individual areas forming the water-shed):

13.—For the type of storm with a uniform (or constant) intensity of rainfall over the entire area throughout the time of concentration of any water-shed, as A_n, with individual areas, $a_1 + a_2 + a_3$, etc. (see Table 1), Conclusion (10) as previously stated, is correct.

The increase of the run-off coefficient ($C_1 < C_2 < C_3$) depends on the previous degree of wetness, as well as the physical characteristics of the water-shed.

[7] Technical Reports, Miami Conservancy Dist., Pt. VIII, "Rainfall and Run-Off in the Miami Valley," by Ivan E. Houk, M. Am. Soc. C. E.

14.—When the expected storm for the time of concentration of any water-shed, as A_n, is of the type that decreases in intensity with the time, there is a tendency toward a balance in Factor C (see Conclusions (10), (11) and (12)). For average rates of retention, on the individual areas, a_1, a_2, a_3, ... a_n, or the areas, A_1, A_2, A_3, ... A_n, varying directly with the respective intensities, then $C'_1 = C'_2 = C'_3 \ldots = C'_n$, or $C_1 = C_2 = C_3 \ldots = C_n$.

Since it is manifestly the intention of the users of the detail process of the rational method to assume a storm of uniform intensity over the entire area throughout the time of concentration, the variable factor of safety (mentioned herein under the heading, "Use of Rainfall Curve"), may be considered as erring from the assumption due to the use of fictitious velocities. In this case Conclusion (13) applies in estimating the run-off from any water-shed, A_n.

However, while the run-off coefficients, C'_1, C'_2, C'_3, etc., correctly apply (under the assumption), to the areas, a_1, a_2, a_3, etc., in the process of estimating the "run-off collecting" at the outlet of the water-shed, A_n, they do not represent the proper values for use in the economical design of the drainage system. For design purposes Areas A_1, A_2, A_3, etc., must be considered as separate water-sheds. For instance, Area A_1, because of its shorter time of concentration, may be expected to receive a higher intensity than Area A_n, and, consequently, a higher run-off coefficient than the value used for it in estimating the run-off for the entire area.

Granting the correctness of Conclusions (10), (11), and (12), therefore, and permitting the use of velocities resulting from the detail method, the users of the customary tabling process, striving for economical design throughout the water-shed, should consider both the magnitude and duration of intensity on the contributing area.

Suggestion (*a*).—In regard to the application of intensities the rational method treats a given drainage area as a separate water-shed above each inlet point in succession on the main channel of flow, the smaller area receiving a higher uniform intensity throughout its time of concentration than the larger, of which it is a part. Those who use a constant run-off coefficient for a given type of territory (Assumption 2), automatically assume, therefore, whether intentional or otherwise, that, as progress is being made on a given rainfall curve, the average rate of retention on each successive water-shed varies directly as the intensity used; that is, that a balance is set up in the coefficient, an increase due to a greater average degree of saturation balancing the decrease due to a lower average intensity.

Stated in another way, the adoption of a constant coefficient (with respect to the time of concentration) for use with a certain rainfall curve, automatically assumes that degree of previous wetness of a given water-shed which will render a balance between the effects of increasing wetness and lowering intensity. Although the rate of retention may vary in different degrees for areas of dissimilar physical characteristics and shape, and the types of rainfall curves differ somewhat, in view of the extremely complicated nature of the problem and the necessary conflicts to be expected in experimental data,

it may be the better part of wisdom at present to set up tentative tables on this basis to be used on the smaller water-sheds and the initial portions of the larger areas (say, the portion contributing during the first 60 to 120 min. of the time of concentration), on which it is desirable to assume an unsaturated surface condition at the outset of the maximum run-off producing intensity.

The making of such tables, among other considerations, would involve the selection of the best average shape; and would necessarily provide different sets of constants, as warranted by the change in average intensity due to frequency periods or location. The 60-min. ordinate could be made the criterion.

The use of such tables would entail the assignment by the designer of values of the coefficient to individual portions of the water-shed on the basis of physical characteristics, the column representing the frequency period of the curve in use having first been determined.

Suggestion (*b*).—For the smaller water-sheds and the portions of the larger areas (say, the portion contributing during the first 60 to 120 min. of the time of concentration), it may be feasible in the light of future experimentation to devise a set of curves (or tables) for elementary areas providing for both the effect of time and intensity on the various soils, degrees of perviousness, etc. It may be reasoned that the use of such curves (for instance, in the detail process) would require the estimation of an average value of the coefficient for the entire area above each inlet as new values of the intensity were selected from the rainfall curve.

Comment.—The uncorrected results of the tabling processes will, in any case, include the automatic error (or factor of safety above the assumption of a uniform intensity over the entire area throughout the time of concentration) noted under the heading, "Use of Rainfall Curve". Where provision is made for consideration of the effect of both time and intensity (as outlined under Suggestions (*a*) and (*b*), for water-sheds comparatively dry at the onset of the maximum run-off producing intensity), a similar error will be automatically included due to the use of an average value of the coefficient for the successive water-sheds above each inlet point, instead of the actual values for the individual areas. The magnitude of this latter error (for water-sheds of uniform characteristics throughout) will depend on the water-shed and the degree of previous wetness assumed, and will be smaller on the average, than that of the former. Both errors are on the side of safety, and, together (considering the approximate nature of the problem), are not of very serious consequence.

It should be evident that, for general use, either in the detail process, or in any method for estimating run-off, a table of values for the run-off coefficient should provide for changing intensities as well as for varying times.

PART II.—RATIONAL RUN-OFF FORMULAS

Derivation of Formulas

In Equation (2) let $t = \frac{L}{V}$, then,

$$i = K \frac{v^e}{L^e} \quad \ldots\ldots (10)$$

Substituting this value of i in Equation (1),

$$Q = C\,A\,K \frac{v^e}{L^e} \quad \ldots\ldots (11)$$

Let $R_H = \frac{X}{H}$, as, for example, $R_{0.5} = \frac{X}{0.5}$, etc. When $i = \frac{K}{t^e}$, $R_H = \frac{K}{(3\,600\,H)^e}$, the time being expressed in seconds; or,

$$K = R_H\,(3\,600\,H)^e \quad \ldots\ldots (12)$$

Velocity may be expressed by the Chezy formula,

$$v = c\sqrt{r\,s} \quad \ldots\ldots (13)$$

Substituting these values of K and v in Equation (11),

$$Q = (3\,600\,H)^e\,C\,A\,R_H \left(c\sqrt{r}\right)^e \frac{S_2^e}{L^e} \quad \ldots\ldots (14)$$

Comparisons.—It should be of interest here to digress temporarily, and convert Equation (14) into the form of the ordinary empirical formula. If S = fall in feet per 1 000 ft. of channel, or $s = \frac{S}{1\,000}$; if the exponent, e, is given the value of 0.5; and if $R_H = R_1$ (the intensity in inches per hour for a period of 1 hour), then Equation (14) becomes,

$$Q = 10.67\,C\,A\,R_1 \left(c\sqrt{r}\right)^{0.5} \frac{\sqrt[4]{S}}{\sqrt{L}} \quad \ldots\ldots (15)$$

The ordinary empirical formula takes no account of the shape of the drainage area; it is usually customary to assume (which later will be shown to be erroneous) that,

$$L = K'\sqrt{A} \quad \ldots\ldots (16)$$

or some constant times $\sqrt{A}$. Assuming, for the purpose of illustration, that Equation (16) is correct, and substituting in Equation (15),

$$Q = 10.67\,K'' \left(c\sqrt{r}\right)^{0.5} C\,R_1 \sqrt[4]{S\,A^3} \quad \ldots\ldots (17)$$

It is interesting here to observe that Equation (17) results in the well-known Bürkli-Ziegler formula when the expression, $10.67\,K'' \left(c\sqrt{r}\right)^{0.5} C$, is considered as a single run-off coefficient. The Hawksley formula may be reduced to similar terms for exponents of S and A, as shown[8] by the late Charles E. Gregory, M. Am. Soc. C. E. Since these and other empirical formulas neglect the shape of the drainage area, do not vary the fluctuating

[8] *Transactions*, Am. Soc. C. E., Vol. LVIII (1907), p. 461.

value of the average velocity, or its function $\left(c\sqrt{r}\right)^e$, and make no allowance for a variation in the nature of the rainfall curve, results accurate enough for general use cannot be reasonably expected from their application.

The value of L is independent of A, not only since the area may be any shape in so far as the exterior boundary is concerned, but also because the water may concentrate to the main channel in an infinite number of ways. To assume, then, that $L = K'\sqrt{A}$, not only presupposes a constant exterior shape of area, but also a constant law of concentration within the area, and is, of course, erroneous.

Introducing Factor F.—It follows, then, that in order to approach accuracy through the medium of a formula, the value of L should be retained, and the effect of velocity in the main channel of flow should be allowed to vary.

In Equation (14), the expression representing varying velocity, as far as shape and condition of the main channel of flow are concerned, is $\left(c\sqrt{r}\right)^e$. At the present time, Kutter's formula for c in $c\sqrt{rs}$ is probably used more widely than any other; and, on account of its general adaptability to all shapes and conditions of channel, it affords a feasible basis for a variable velocity factor in run-off formulas. Since the velocity varies inversely as the time, the velocity factor, $c\sqrt{r}$, in a run-off formula, carries the exponent of the time, e. Due to the construction of rainfall curves this exponent varies in practice from about 0.4 to 0.6, the average probably being approximately 0.5.

In order to visualize more easily the effect of the fluctuating value of $c\sqrt{r}$ on run-off, the writers made many listings from "Hydraulic and Excavation Tables", published by the United States Reclamation Service. The process consisted of assuming first, a definite shape of channel, such as a circular pipe, or a closed rectangular box with a constant ratio of width to height. For a definite shape of channel on a given slope and Kutter's n, the hydraulic radius was allowed to vary, the corresponding velocity was selected from the hydraulic tables, and the values of Q, area, $c\sqrt{r}$, and $\left(c\sqrt{r}\right)^{0.5}$, were calculated.

It was found that for any one definite shape of channel on a given slope and Kutter's n, $\left(c\sqrt{r}\right)^{0.5}$, divided by $Q^{\frac{1}{8}}$, remained practically constant for all values of $c\sqrt{r}$ greater than 50 to 60, or,

$$\left(c\sqrt{r}\right)^{0.5} = N\,Q^{\frac{1}{8}} \qquad (18)$$

in which, N represents a constant. More recent experimentation has shown that Kutter's formula is somewhat in error for very small sections. In selecting values of N (obtained by solving Equation (18)), choose from the Hydraulic Tables of the Reclamation Service, hydraulic radii rendering values of $c\sqrt{r}$ greater than 60; velocities may then be calculated, for the smaller sections, which are closer on the average to recent experimentation than those given by the use of Kutter's formula. In any case Equation (18) offers a reasonably accurate basis for obtaining a variable velocity factor, representing the shape and condition of the main channel of flow, for run-off formulas. The method of arriving at values of N and F is shown in more detail in the Appendix II.

As shown in Equation (14), $Q = (3\,600\ H)^e\ CAR_H \left(c\sqrt{r}\right)^e \frac{S^{\frac{e}{2}}}{L^e}$; but, hydraulically, $\left(c\sqrt{r}\right)^{0.5} = N\ Q^{\frac{1}{8}}$, or $c\sqrt{r} = N^2\ Q^{0.25}$. It must be remembered that the term, $c\sqrt{r}$, occurring in Equation (14), represents the average value of $c\sqrt{r}$ as determined by the average velocity of the water in running from the most remote part of the water-shed to its outlet. In this equation then, let $c\sqrt{r} = P\ N^2\ Q^{0.25}$, or,

$$\left(c\sqrt{r}\right)^e = P^e\ N^{2e}\ Q^{\frac{e}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Methods of arriving at values of P are discussed subsequently in this paper.

Substituting $s = \frac{S}{1\,000}$, and the value of $\left(c\sqrt{r}\right)^e$ shown in Equation (19), into Equation (14),

$$Q = \frac{(3\,600\ H)^e}{(1\,000)^{\frac{e}{2}}}\ CAR_H\ P^e\ N^{2e}\ Q^{\frac{e}{4}}\ \frac{S^{\frac{e}{2}}}{L^e} \quad \ldots\ldots\ldots\ldots\ldots (20)$$

From Equation (18), $N = \frac{\left(c\sqrt{r}\right)^{0.5}}{Q^{\frac{1}{8}}}$, that is, $N \propto \frac{1}{Q^{\frac{1}{8}}}$; but $Q =$ area $\times \left(c\sqrt{r\,s}\right)$. Therefore, Q, for a given channel, may be considered to vary directly as the square root of s, since the value of c varies only slightly with a change in slope. Then, it may be written that $Q \propto S^{\frac{1}{2}}$; and, since $N \propto \frac{1}{Q^{\frac{1}{8}}}$, $N \propto \frac{1}{S^{\frac{1}{16}}}$, or, $N = \frac{F}{S^{\frac{1}{16}}}$, in which, F represents a constant (see Appendix II).

Then,

$$N^{2e} = \frac{F^{2e}}{S^{\frac{e}{8}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Substituting this value of N^{2e} in Equation (20),

$$Q = \frac{(3\,600\ H)^e}{(1\,000)^{\frac{e}{2}}}\ CAR_H\ P^e\ Q^{\frac{e}{4}}\ F^{2e}\ \frac{S^{\frac{3e}{8}}}{L^e} \quad \ldots\ldots\ldots\ldots\ldots (22)$$

$$Q^{1-\frac{e}{4}} = \frac{(3\,600\ H)^e}{(1\,000)^{0.5e}}\ CAR_H\ P^e\ F^{2e}\ \frac{S^{\frac{3e}{8}}}{L^e} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

or, if the exponent, $\frac{1}{4-e} = g$:

$$Q = \frac{(3\,600\ H)^{4eg}}{(1\,000)^{2eg}} \left(\frac{P}{L}\right)^{4eg} (CAR_H)^{4g}\ F^{8eg}\ S^{1.5eg} \quad \ldots\ldots\ldots (24)$$

Equation (24) is a general expression for the value of Q with seven main factors: C, A, R_H, S, P, F, and L. Factor C, of course, should be determined from experimental information, while A, S, and L must be derived from topographical maps or surveys.

Factor F varies with the shape and condition of the main channel of flow. Values for this factor may be selected from Table 10 in Appendix II. Factor P may be selected from Tables 3 and 4. The method used for the derivation of the values in these tables is discussed in another part of this paper.

Factor R_H is necessarily involved with H and e. The combination renders flexibility in the choice of the time period for observation of rainfall data, the type of rainfall curve, and storms of various expected frequency periods. Thus, for the smaller areas, it may be found advisable to choose 1 hour as the guiding period for rainfall observation, in which case, $H = 1$ hour, and R_H is represented by R_1. For the larger areas circumstances may make more feasible the adoption of longer time periods, as $H = 6, 12, \ldots 24$, with $R_H = R_6$, R_{12}, or R_{24}, as the case may be. Thus, when $R_H = \frac{X}{H}$, the expected rainfall may be $4\frac{1}{2}$ in. for a period of 6 hours. The average expected rainfall intensity expressed in inches per hour, for a 6-hour period, equals $R_H = \frac{X}{H} = \frac{4.5}{6.0} = 0.75$. The exponent, e, of the time in Equation (2) determines the type of rainfall curve.

Inadequacy of Empirical Formulas

At this point the reader may gain, by comparison, a more thorough comprehension of the inadequacy of the leading formulas of the empirical type, the construction of which offers opportunity for variation only in Factors C, A, R, and S. This comparison may be clarified by five statements as follows:

(a) Factor R_H of Equation (24) is represented in the empirical formula by R. The factor, R, in the empirical formula is usually assumed to represent the expected average rainfall intensity, in inches per hour, for a 1-hour period. This is, of course, a specialization of R_H.

(b) The exponent, e, of Equation (24) is necessarily given a constant value in any one empirical formula, thus rendering inflexibility in the choice of the type of the rainfall curve.

(c) Factor P of Equation (24), is, in any one empirical formula, given a constant value. The reader may gain some conception of the range of variation in this factor by a study of Tables 3 and 4.

(d) In the same manner, Factor F of Equation (24) is represented in any one empirical formula by some constant. Opportunity for measuring the effect on run-off, due to variation in shape and condition of the main channel of flow, is thereby rendered impossible. (See Table 10, Appendix II.)

(e) In the empirical formulas, in which the length of the main channel of flow, L, does not appear, the effect on run-off due to the shape of the area cannot be measured. The absence of L entails the assumption that it equals some constant times A to an exponent, as $L = K' \sqrt{A}$, thereby precluding the possibility of weighing the effect on run-off caused by varying the shape of the water-shed either in external boundary or manner of internal concentration.

Reduction of Equation (24)

This equation reduces to simpler terms, once the required data on rainfall are known. For example, should the rainfall data for a given locality point to the use of a curve following the equation, $i = \frac{K}{t^{0.6}}$, and should it be found advisable to use a 12-hour time period, then $H = 12$ and $e = 0.6$.

Substituting these values, Equation (24) reduces to:

$$Q = 163.5 \left(\frac{P}{L}\right)^{0.706} (CAR_{12})^{1.176} F^{1.412} S^{0.265} \ldots\ldots\ldots\ldots (25)$$

Thus, when H and e are once adopted, Equation (24) reduces to the seven main factors with definite exponents, prefixed with a constant. Where a formula of the type of Equation (25) is constantly in use, it probably would be found advisable to reduce it to four variables and solve by nomograph in two parts, such that $\left(\frac{P}{L}\right)^{0.706} \times (CAR_H)^{1.176}$ forms one nomograph, and $F^{1.412} S^{0.265}$, another. The formula also lends itself to quick solution by log-log slide-rule.

To find the time of concentration compute the value of i from Equation (1). For this purpose, Q is determined from Equation (24) reduced. The time, t, may then be read from a curve of Equation (2). If there is no diagram, t, in minutes, may be computed from Equation (2) by substituting for K its value in Equation (12) and dividing the result by 60; thus,

$$t = \frac{60\, H\, R_H^{\frac{1}{e}}}{i^{\frac{1}{e}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Special Reduction of Equation (24).—It is realized that occasions may arise necessitating the estimation of run-off in territories where adequate rainfall records are not available. Even in localities where available records vary somewhat from average conditions, it may be found unnecessary, or economically inadvisable, to go into great refinement in the making of rainfall curves.

Mr. Grunsky[9] and others advocate the use of a rainfall curve of the form of Equation (3). Moreover, the writers have found that it applies very well to conditions in Southern California, as, for example, it conforms very closely to the curves now used by the City of Los Angeles, the Los Angeles County Flood Control Department, and the Los Angeles County Storm Drain Division. The writers, therefore, have developed the following special reduction with a twofold purpose in mind, namely, (*a*), to furnish a formula which will be approximately applicable for general use; and (*b*), one which may very readily be used for making quick transitions to other types of curves as subsequently recommended.

In Equation (24) if $e = 0.5$, $g = 0.2857$, and,

$$Q = 15\, H^{0.5714} \left(\frac{P}{L}\right)^{0.5714} (CAR_H F)^{1.1429} S^{0.2143} \ldots\ldots\ldots (27)$$

[9] *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 67.

and since the expression, $\left(\frac{P}{L}\right)^{0.5714} = \left(\sqrt{\frac{P}{L}}\right)^{1.1429}$, Equation (27) may be written:

$$Q = 15\ H^{0.5714}\ (CAR_H\ F\ B)^{1.1429}\ S^{0.2143} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

in which, $\sqrt{\frac{P}{L}} = B$, which may readily be obtained by slide-rule.

Equation (28) reduces to two variables if the time period, H, is adopted, and the R_H value is determined from the relation, $R_H = \frac{X}{H}$. Hence, when $H = 1$ hour, Equation (28) reduces to:

$$Q = 15\ (CAR_1\ F\ B)^{1.1429}\ S^{0.2143} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

Whenever the adopted rainfall curve varies materially from Equation (3), use may be made of Equation (28) to give an approximate value of Q to be corrected as follows:

Substitute the values of Q, C, and A, of Equation (28) in Equation (1) and solve for i. Then obtain the time, t, in minutes, from a diagram of Equation (3), or by solving Equation (26). Using this value of t, obtain a corresponding value of i from the curve in use. With this value of i solve Equation (1) for a corrected Q, which will be a close approximation to the quantity of run-off to be expected from the use of the adopted rainfall curve.

Method of Arriving at Values of Factor P

Factor P has been defined as the ratio of the average velocity of flow to velocity at the outlet of the water-shed, based on average values of S and F or $P = \frac{v}{V}$. Upon inspection of Equation (24), reduced to its seven main factors—C, A, R_H, S, P, F, and L—it may be readily seen that when average values for a given water-shed are assigned to Factors S, F, C, and R_H (these factors then assume the rôle of constants in the formula), the variation in Factor P is determined by the remaining two factors, A and L. The fluctuation in Factor P, therefore, depends on the shape of the drainage area, both as to external boundary and internal concentration.

It is evident, of course, that P may have an infinite number of values. Since, in any rational method of estimating run-off, inlet time must be considered, the value of P is further complicated. Factor $P = \frac{v}{V}$; but $v = \frac{L}{t}$, in which, t = time of concentration of the water-shed, including the initial, or inlet, time of the first small area, or nucleus.

Before attempting to devise a table of values for Factor P it was necessary to investigate both the use of the rainfall curve and the run-off coefficient (particularly as regards its variation with the time). The reader at this point is assumed to be familiar with the conclusions under headings, "Use of Rainfall Curve," and "Factor C, The Run-Off Coefficient." For a given area it may be reasoned that:

(*a*) The use of a factor, C, increasing with the time would give a value of P somewhat lower than a constant C.

(*b*) The use of a factor, C, decreasing with the time would give a value of P somewhat higher than a constant C.

(*c*) The type of storm assumed for a given water-shed has a certain effect on the expected run-off. This effect influences not only the run-off coefficient, but also Factor P. If, for a given area, it is assumed that the storm for the time of concentration decreases in intensity with the time, a somewhat higher value of P results than that for the assumption that the storm is one of uniform intensity throughout.

The effects on Factor P of the foregoing considerations under Paragraphs (*a*), (*b*), and (*c*), are, however, not of major importance, since (except for extreme conditions) the final run-off is varied only a few per cent. one way or the other. The latitude for variation in the run-off result, due to errors in judgment in choosing values for Factor C, R_H, and F, is large in comparison; but, on account of the many varying conditions entering into the estimation of run-off by any method, the engineer, as far as practicable, should be able to visualize all limitations or advantages of any particular formula or system.

Estimation of run-off inevitably depends to a great extent on individual judgment. The writers, in evolving formulas, are merely attempting to aid this judgment in a quick solution. In the face of so many variations it would be highly impracticable to attempt a mathematical solution of P for the purpose of making the tables of values for use in the formulas. It is possible, however, to establish workable tables by trial method, that should be far superior at least to the use of a constant (single) value of this factor, as is automatically done when an empirical formula is used. An average, or constant, value of C was used in preparing Tables 3 and 4. The values in Table 3 are based on the assumption of a storm of uniform (or constant) intensity for the time of concentration. Hence, a constant intensity throughout was used on any one area type shown in Fig. 2. Table 3, therefore, should be used when the assumption is that the expected storm is one of uniform intensity for the time of concentration. Referring to Conclusion (13) under "Factor C, The Run-Off Coefficient" (Part I), it may be seen that the use of this table carries with it a small factor of safety, which is comparable with, but smaller than, that automatically assumed in the usual detail process of the rational method (see "Statement" following Conclusion (4)); and its use, rather than being objectionable, is to be commended, especially when the possible variation in types of storms is taken into consideration.

The values in Table 4 were evolved through the use of a rainfall curve of the type of Equation (3), in the manner ordinarily used in the detail process of the rational method. This table, therefore, involves the small factor of safety resulting from the use of Equation (3) as ordinarily practiced in the detail rational method.

Values of P in Tables 3 and 4 may be used in any formula reduced from Equation (24). In preparing Tables 3 and 4, the procedure was as follows:

1.—Areas, shown on Fig. 2, were chosen for the purpose of establishing practical limits of shape, so that interpolations could be made for values of P.

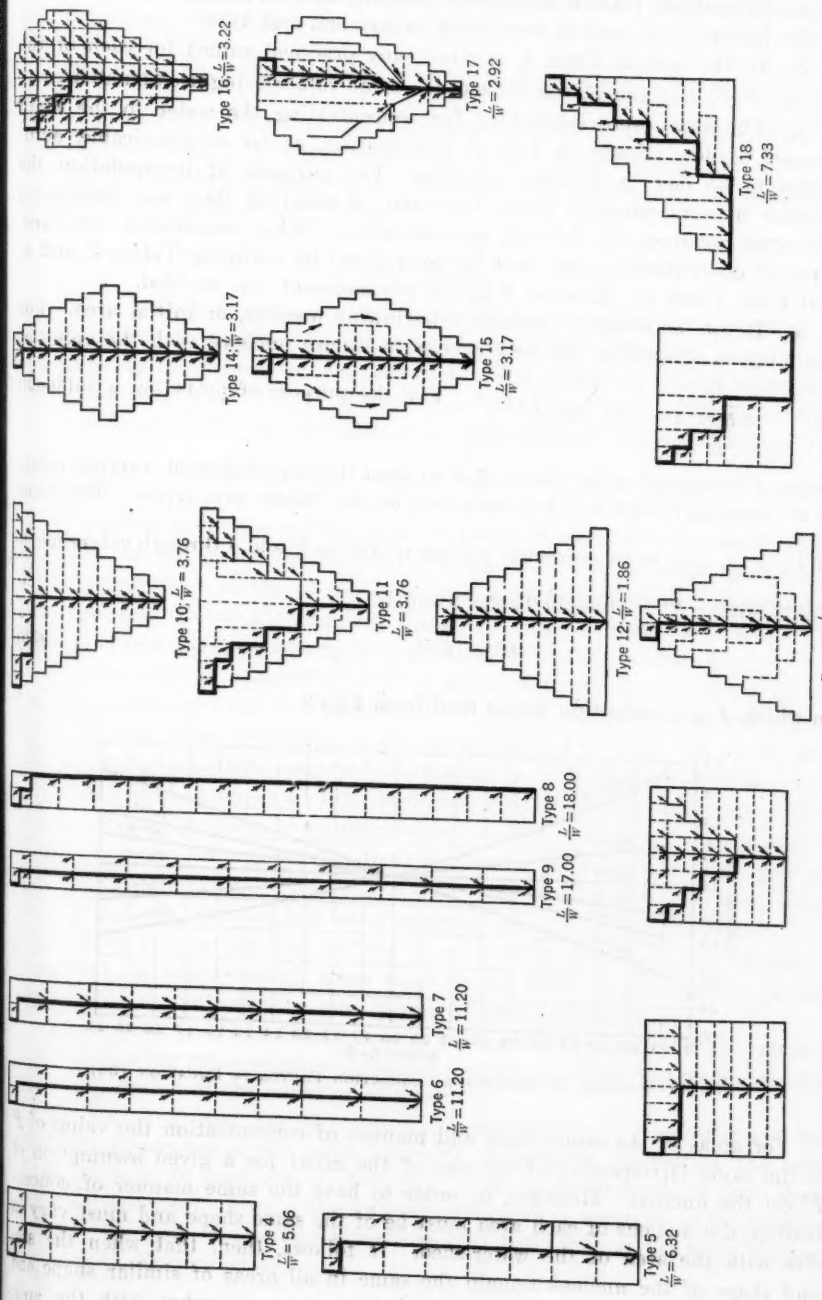


FIG. 2.—TYPICAL AREAS AND THEIR MANNER OF CONCENTRATION

2.—In making Table 3, a constant intensity and an average value for each of the factors, S, C, and F, were used on any one area type.

3.—In the case of Table 4, a value (any practical value) for each of the factors, S, C, R_H, and F, was chosen, the same values being used on all areas.

4.—The areas were subdivided for concentrating the water to the main channel of flow (shown on Fig. 2) anticipating, as far as practicable, actualities to be met in drainage practice. For purposes of interpolation the manner of concentration along the main channel of flow was designated advanced, medium, or delayed, concentration. What constitutes the three types of concentration may best be ascertained by studying Tables 3 and 4, and Figs. 2 and 4. Extreme delay or advancement was avoided.

5.—It was necessary to assume a beginning nucleus, or initial area. For the purpose of striking the best average a square nucleus, with the value of $\frac{L'}{W'} = \frac{L'}{\frac{43\,560\,A'}{L'}} = \frac{(L')^2}{43\,560\,A'} = 4$. For the purpose of providing a sufficient range of interpolation for values of P to meet the expected need, varying values of P', ranging from 0.1 to 2.0, were used on the various area types. The value of $P_1 = \frac{v'}{V'}$, is to be corrected for use in Tables 3 and 4 through reference to Fig. 3, giving P'. Thus, in Tables 3 and 4,

$$P' = f\,P_1 = f\,\frac{v'}{V'} \qquad (30)$$

in which, f is a correction factor read from Fig. 3.

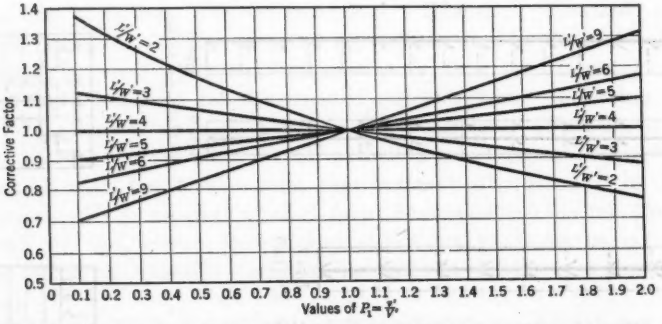


FIG. 3.—DIAGRAM TO DETERMINE CORRECTION FACTOR IN EQUATION (30).

For areas of the same shape and manner of concentration the value of P is the same (irrespective of the size of the area) for a given assumption of P' on the nucleus. However, in order to have the same manner of concentration, the nucleus of each area must be of the same shape and must vary in size with the area of the water-shed. It follows, then, that when the size and shape of the nucleus remain the same in all areas of similar shape and manner of concentration, the value of P changes somewhat with the area.

In Tables 3 and 4 a constant nucleus was used; hence, the variation in P with the area divided by area of nucleus as shown.

The ratio, $\frac{L}{W}$, in these tables was adopted as a practical means of helping to define the change in area shape when used in conjunction with the typical areas shown in Fig. 2, which are also intended for guides as to the manner of internal concentration. The ratio, W, equals area of water-shed, in square feet, divided by L. Hence,

$$\frac{L}{W} = \frac{L^2}{43\,560\,A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Fig. 4 will be found useful in the determination of delayed, medium, or advanced concentration. This diagram is intended to be used as an aid in determining the manner of concentration. The numbered points refer to the area types shown on Fig. 2. The relative positions of the various points for a given value of P' were determined by using values of P for abscissas (not shown on the diagram) to attain compactness, and $\frac{L}{W}$ for ordinates.

P' having been determined, and the area type on Fig. 2 (most nearly representing the area being studied) selected, enter Fig. 4 under the value of P' and note the position of the area type with respect to the concentration lines marked "Delayed", "Medium", and "Advanced"; then enter Table 3 or Table 4 under the heading of the concentration chosen on Fig. 4.

The manner of concentration of a given water-shed, designated in this work, delayed, medium, or advanced, is, of course, determined by the relative quantities of water gathering to the main channel of flow. In small areas these relative quantities may be contributed by individual subdivisions without well-defined channels of flow (such as a city block); in larger water-sheds these contributions may be represented by laterals; and in still larger ones by creeks, or rivers, as may be visualized by a glance at a topographical map of a large mountainous area. Hence, the area types on Fig. 2 may be used for an area of any size, the sketches being intended only to aid the judgment. It is realized, of course, that the method is not perfect, and that it is capable of expansion or improvement, since there is no limit to the number of area shapes; yet the writers hope that it may at least be helpful.

Although it would be impractical to solve mathematically for all values of P, the following two analytical solutions may serve to help the reader in forming a conception of the factor's usefulness.

Let Fig. 5(a) and Fig. 5(b) represent water-sheds of triangular and rectangular shapes, as shown, each subjected to uniform intensities, the water being assumed to concentrate uniformly to the main channels of flow, designated by the heavy lines. The run-off coefficient is assumed to be constant, and the velocities (combining Equations (13) and (18)) in the main channels of flow are determined by,

$$v = N^2\, Q^{\frac{1}{4}}\, S^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

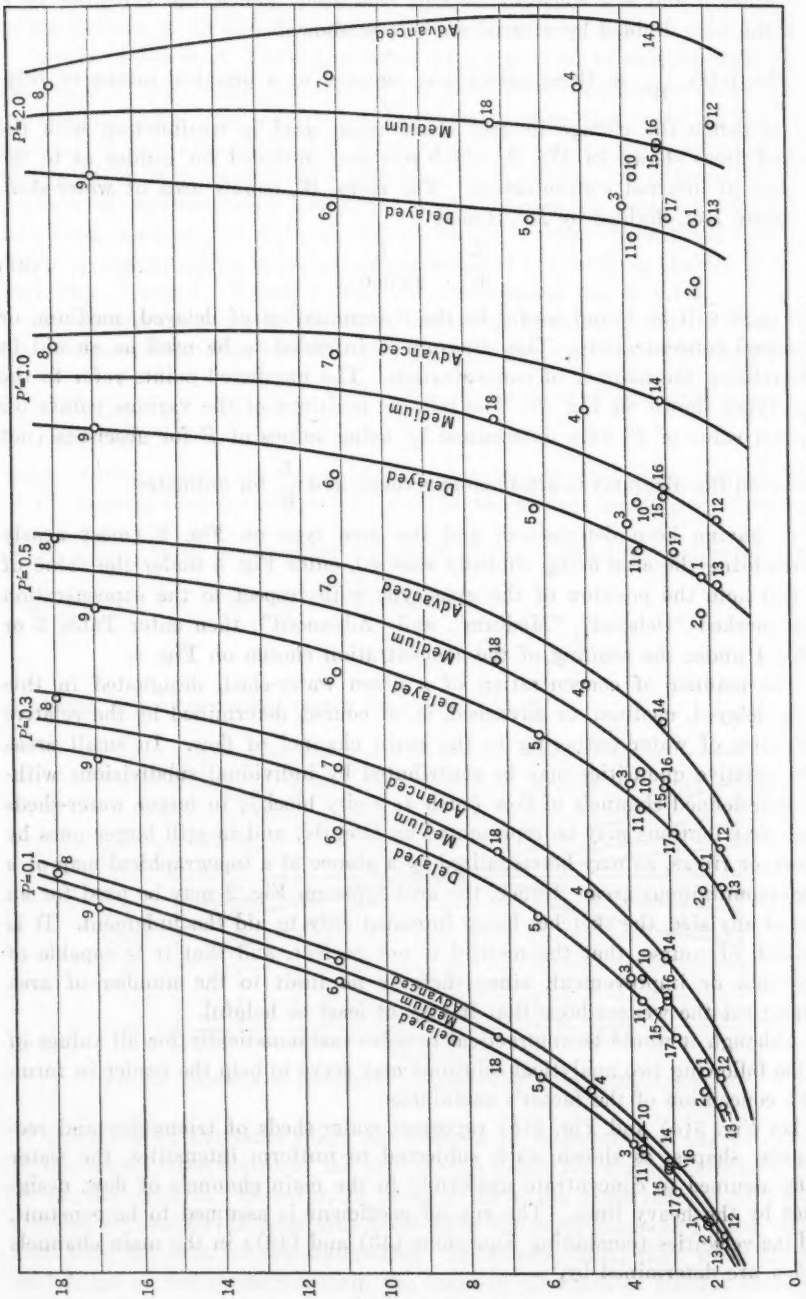


FIG. 4.—GUIDE LINES TO AID IN DETERMINATION OF THE MANNER OF CONCENTRATION

Example 1.—The Triangular Water-Shed.—In Fig. 5(*a*) the shape of the water-shed above any point on the main line is the same; that is, $L \propto \sqrt{A}$, or $A \propto L^2$.

Under the foregoing assumptions the run-off, Q, concentrating to points on the main line, varies directly as the contributing area, or, $Q \propto A$; and, hence, $Q \propto L^2$.

Since $v \propto Q^{\frac{1}{4}}$ (see Equation (32)), $v \propto (L^2)^{\frac{1}{4}} \propto L^{\frac{1}{2}}$. Therefore, $v \propto t$, or,

$$v = M\,t \qquad (33)$$

in which, M represents some constant.

The average value of v may be found as follows: When $v = M\,t$, the expression for the area, $v\,t\ (= L)$, is $\int M\,t\,dt$, or $\frac{1}{2}\,M\,t^2$; and $\frac{\frac{1}{2}\,M\,t^2}{t} = \frac{1}{2}\,M\,t$, or the average velocity, is one-half the maximum velocity. Thus, the value of $P = \frac{v}{V}$, under the foregoing assumptions, is one-half.

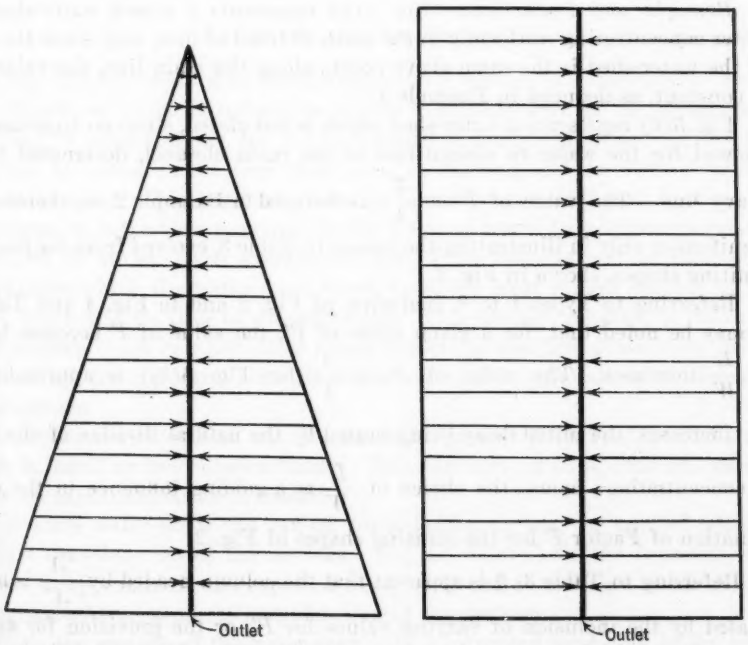


FIG. 5.—DIAGRAMS TO INDICATE THE USEFULNESS OF THE FACTOR P

Example 2.—The Rectangular Water-Shed.—The value of P for this shape is an impractical one, because the time of concentration of the water to the main line can not be taken into account; but its determination is of value for purposes of illustration and comparison. In Fig. 5(*b*) it may be seen by

FIG. 4.—GUIDE LINES TO AID IN DETERMINATION OF THE MANNER OF CONCENTRATION

inspection that the length, L (measured from the top of the water-shed to any point on the main line), varies directly as the contributing area to that point, or $L \propto A$; and since (as before), $Q \propto A$, then $Q \propto L$. Since $v \propto Q^{\frac{1}{4}}$, $v \propto L^{\frac{1}{4}}$. Furthermore,

$$t = \frac{L}{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

and, hence, $t \propto \frac{L}{L^{\frac{1}{4}}} \propto L^{\frac{3}{4}}$, or, $L \propto t^{\frac{4}{3}}$; therefore, since $v \propto L^{\frac{1}{4}}$, $v \propto \left(t^{\frac{4}{3}}\right)^{\frac{1}{4}} \propto t^{\frac{1}{3}}$, or,

$$v = M' t^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

Therefore,

$\int M' t^{\frac{1}{3}} dt = \frac{3}{4} M' t^{\frac{4}{3}} =$ area, $v t$ $(= L)$, and $\frac{\frac{3}{4} M' t^{\frac{4}{3}}}{t} = \frac{3}{4} M' t^{\frac{1}{3}} =$ the average velocity. Thus, the average velocity is three-fourths the maximum velocity, or $P = \frac{v}{V} = \frac{3}{4}$.

Remarks and Deductions.—Fig. 5(a) represents a closed water-shed, the water concentrating uniformly to the main channel of flow, and, since the shape of the water-shed is the same above points along the main line, the value of P is constant, as deduced in Example 1.

Fig. 5(b) represents a water-shed which is not closed, since no time has been allowed for the water to concentrate to the main channel, designated by the heavy line. The value of $P = \frac{3}{4}$, as deduced in Example 2, is, therefore, of significance only in illustrating the values in Table 3, evolved from the practical limiting shapes, shown in Fig. 2.

Referring to Types 1 to 9, inclusive, of Fig. 2 and to Fig. 4 and Table 3, it may be noted that, for a given value of P', the value of P becomes higher as $\frac{L}{W}$ increases. The value of $P = \frac{3}{4}$ (for Fig. 5(b)) is approached as $\frac{L}{W}$ increases, the initial delay being caused by the natural division of the water in concentrating; hence, the choice of $\frac{L}{W}$ as a guiding influence in the determination of Factor P for the limiting shapes of Fig. 2.

Referring to Table 3, it is apparent that the column headed by $\frac{A}{A'}$ is necessitated by the inclusion of varying values for P', or the provision for assignment of an initial time for the inlet area. This provision was made in order to create possibilities for the broadest use of the formulas. It is evident, however, that for many water-sheds, especially the larger ones, the consideration of the inlet area as a separate entity, may be unnecessary. In this case, the water-shed is neither delayed, nor advanced by the inlet areas, and its condition at the outset of the maximum run-off producing intensity is ready for the falling water to begin concentrating immediately.

When the inlet area is not considered separately, the slope, S, of the water-shed must be based on the total slope from the most remote area (including the nucleus) to the outlet, and the value of $\frac{A}{A'}$ in Table 3 must be considered as infinite.

As $\frac{A}{A'}$ increases (see Table 3), the values of P decrease under the columns headed by the higher values of P' and increase for the lower values of P'. The natural convergence of these values as $\frac{A}{A'}$ approaches infinity is approximately to the column headed by $P' = 0.5$.

For all practical purposes, therefore, the values of P (in Tables 3 and 4) shown opposite $\frac{A}{A'} = 262\ 144$, and in the column headed by $P' = 0.5$, may be used when the formulas are applied without separate consideration of the inlet area, or nucleus.

The Inlet Area, or Nucleus

In any rational method of estimating run-off special consideration in many cases should be given to the choice of the beginning area of the water-shed. The time of concentration for the water-shed includes the time required for the water to gather in the first small area, or nucleus, and travel to a defined channel of flow. This beginning area is a small water-shed and its time of concentration is the time required for the water to travel from the most remote portion to its outlet. The average velocity of the nucleus may be defined as the distance which the water must travel from the most remote portion to the outlet, divided by its time of concentration. This average velocity, in the same small nucleus (such as a city block), may consist of flow over grass plots, driveways, and gutters; in rural areas, it will be retarded by vegetation, or increased by barrenness.

It is customary for those who use the detail process to assume an inlet time, which is based on field observations. Reliable data on inlet time are scarce, and more experimentation is needed.

For some water-sheds the run-off formulas herein may be applied without separate consideration of the nucleus. For all water-sheds the formulas may be used to approximate the expected value of Q and to estimate times of concentration without separate consideration of the nucleus.

When the surface of the water-shed is in a saturated condition at the onset of the maximum run-off producing intensity, the inlet time of concentration for a high intensity will be shorter than that for lesser ones, due only to the greater quantity of water flowing. If the surface is not saturated the time of concentration for the greater intensities will be shorter than for lesser ones, due both to the greater quantity of water flowing and to a shorter period required for saturating the surface sufficiently for run-off to occur.

In any method of computing run-off great care should be used in choosing the main channel of flow on which the time of concentration for the water-shed so largely depends. The run-off, Q, sought for a given water-shed is the maximum that may be had for definite assumptions on rainfall. For some areas, with extreme contrasts in values of characteristics governing the run-off coefficient, and slope, shape, and condition of the main channel of flow, a part of a given water-shed may be found to give a greater run-off than the whole.

Experience is of value in the choice of the main channel of flow; but, even so, it may sometimes be found necessary to determine it by trial.

The initial area, or nucleus, is a part of the water-shed (and the length, L, of the water-shed includes L' of the nucleus). Consequently, the choice of the nucleus plays no small part in the determination of the maximum run-off. A good conception of the influence of the nucleus may be had from a study of Tables 3 and 4.

Average Slope

Heretofore, in using formulas for the estimation of run-off, the writers have been unable to determine a practical method for calculating the average slope. Of course, the slope should be that of the main channel of flow, that is, the channel which determines the time of concentration of the water-shed.

The average hydraulic slope is not the average of the various slopes weighted by the individual lengths, the sum being divided by the total length as, $S = \frac{1}{l}\ (L_1 S_1 + L_2 S_2 + L_3 S_3 \text{ etc.})$, in which, l equals the sum, $L_1 + L_2 + L_3$, etc. By inspection it may be seen that this average slope is identical with $S = \frac{\text{total fall}}{\text{total length}}$, which would be easily obtained.

Average hydraulic slope depends on the velocities induced by the individual slopes. The average velocity, therefore, will be induced by the true average hydraulic slope. Since the average velocity equals the total length divided by the time, or $v = \frac{l}{t}$, it may be seen that the true average hydraulic slope is that which, with a given quantity of water flowing through length, l, will cause the same total length of time to elapse as that consumed in traveling the slopes, S_1, S_2, S_3, etc.

In Equation (32), $v = N^2\, Q^{\frac{1}{4}}\, \sqrt{s}$, or $v \propto N^2 \sqrt{S}\, Q^{\frac{1}{4}}$. Since $N = \frac{F}{S^{\frac{1}{6}}}$ (according to Equation (21)), $v \propto F^2\, S^{\frac{3}{8}}\, Q^{\frac{1}{4}}$; but the time, t, varies as length divided by velocity, or $t \propto \frac{l}{v} \propto \frac{l}{F^2\, S^{\frac{3}{8}}\, Q^{\frac{1}{4}}}$. It follows, therefore, that if the average hydraulic slope is to cause the same lapse of time as that consumed by the water in traversing the individual slopes, S_1, S_2, S_3, etc.,

$\frac{1}{S_1^{\frac{3}{8}}}, \frac{1}{S_2^{\frac{3}{8}}}, \frac{1}{S_3^{\frac{3}{8}}}$, etc., should be weighted by $\frac{L_1}{F_1^2\ Q_1^{\frac{1}{4}}}, \frac{L_2}{F_2^2\ Q_2^{\frac{1}{4}}}, \frac{L_3}{F_3^2\ Q_3^{\frac{1}{4}}}$, respectively. For example,

$$\frac{1}{S^{\frac{3}{8}}} = \frac{\frac{L_1}{F_1^2\ Q_1^{\frac{1}{4}}\ S_1^{\frac{3}{8}}} + \frac{L_2}{F_2^2\ Q_2^{\frac{1}{4}}\ S_2^{\frac{3}{8}}} + \text{, etc.}}{\frac{L_1}{F_1^2\ Q_1^{\frac{1}{4}}} + \frac{L_2}{F_2^2\ Q_2^{\frac{1}{4}}} + \text{, etc.}}$$

or, inverting,

$$S^{\frac{3}{8}} = \frac{\frac{L_1}{F_1^2\ Q_1^{\frac{1}{4}}} + \frac{L_2}{F_2^2\ Q_2^{\frac{1}{4}}} + \text{, etc.}}{\frac{L_1}{F_1^2\ Q_1^{\frac{1}{4}}\ S_1^{\frac{3}{8}}} + \frac{L_2}{F_2^2\ Q_2^{\frac{1}{4}}\ S_2^{\frac{3}{8}}} + \text{, etc.}} \quad \ldots\ldots\ldots\ldots (36)$$

Thus, $S^{\frac{3}{8}}$, and then S, the average slope, may be solved.

Example 3.—F and Q Constant.—Either s or S may be used throughout; F and Q, being constant, drop them from Equation (36) and assume $L_1 =$ 1 848 ft.; with $S_1 = 90$, and $L_2 = 8\,316$ ft., with $S_2 = 0.9$ then (computations by slide-rule) substituting these values in Equation (36), $S^{\frac{3}{8}} = 1.131$; and $S = 1.39$ ft. per 1 000 ft.

A check on this answer is as follows: Assume that $Q = 200$ cu. ft. per sec., and that $F = 6.63$, then by Equations (21) and (32),

$$v = N^2 \sqrt{s}\ Q^{\frac{1}{4}} = \frac{F^2}{S^{\frac{1}{8}}} \sqrt{s}\ Q^{\frac{1}{4}} \quad \ldots\ldots\ldots\ldots\ldots (37)$$

Substituting the assumed values in Equation (37), $V_1 = 28.3$ ft. per sec.; and $t_1 = \frac{L_1}{v_1} = \frac{1848.0}{28.3} = 65$ sec.

Similarly (see Equation (37)), $v_2 = \frac{44.0}{0.988} \times 0.03 \times 3.76 = 5.0$ ft. per sec.; $t_2 = \frac{8\ 316}{5.0} = 1\,655$ sec.; and $t = t_1 + t_2 = 65 + 1\,655 = 1\,720$ sec. (actual time).

Using the average slope, $S = 1.39$, in Equation (37), $v = 5.91$ ft. per sec.; and $t = \frac{l}{v} = \frac{10\ 164}{5.91} = 1\,720$ sec., which checks the time on individual slopes.

Example 4.—F Constant and Q Varying.—In this case, only F drops from Equation (36). Assume that $L_1 = 1\,848$ ft., with $S_1 = 90$ and $Q_1 = 10$ cu. ft. per sec., and $L_2 = 8\,316$ ft., with $S_2 = 0.9$ and $Q_2 = 656$ cu. ft. per sec. Then, substituting in Equation (36) as before, $S^{\frac{3}{8}} = 1.41$, and $S = 2.50$ ft. per 1 000 ft.

As a check, assume that $F = 6.63$, then, by Equation (37): $v_1 = \frac{44.0}{1.755} \times 0.3 \times 1.776 = 13.36$ ft. per sec., and $t_1 = \frac{1\ 848}{13.36} = 138$ sec. Similarly,

(see Equation (37)), $v_2 = 6.76$ ft. per sec. and $t_2 = 1\,230$ sec. As in Example 3, $t = t_1 + t_2 = 138 + 1\,230 = 1\,368$ sec. (actual time).

Using the average slope, $S = 2.50$ in Equation (37), $v_1 = 3.49$ ft. per sec., and $t_1 = \frac{1\,848}{3.49} = 530$ sec. Likewise, $v_2 = 9.94$ ft. per sec., and $t_2 = \frac{8\,316}{9.94} = 837$ sec. As before, $t = t_1 + t_2 = 530 + 837 = 1\,367$ sec., which checks the total time on the individual slopes.

Example 5.—F and Q Varying.—In this case, Equation (36) is used directly. For instance, assume that $L_1 = 9\,240$ ft., with $S_1 = 90$, $F_1 = 5.0$, and $Q_1 = 200$ cu. ft. per sec., and $L_2 = 7\,392$ ft., with $S_2 = 0.9$, $F_2 = 7.0$, and $Q_2 = 800$ cu. ft. per sec.

Substituting the proper values in Equation (36), $S^{\frac{3}{8}} = 2.65$ and $S = 13.52$ ft. per 1 000 ft. A check on this average value of S may be made as in Examples 3 and 4.

Practical Use of Method.—The foregoing method of estimating the true average hydraulic slope will be found useful, no doubt, in general hydraulic work. It will be found especially useful in run-off formulas when the slope of the main line of flow of the water-shed changes materially.

In the run-off formulas the exponent of the slope is, of course, determined by the type of rainfall curve. For instance, when the curve is of the type of Equation (2), and e has the value 0.4, 0.5, or 0.6, there results approximately $S^{0.167}$, $S^{0.214}$, $S^{0.265}$, respectively, for use in the run-off formulas. On account of the smallness of these exponents in the average formula, extreme refinement in estimating the average slope is unnecessary. However, the extreme differences in slopes of sections of the main channel, encountered in many water-sheds, require a rational approach to the true average slope. The refinement resorted to should depend both on these differences and on the requirements of the particular problem at hand. The water-shed may be divided into two, or more, parts, according to the judgment of the engineer.

When the expected storm is assumed to have a uniform (or constant) intensity for the time of concentration, Q varies directly as the product, CA. It may be seen, therefore, that, for this assumption, the product, CA, may be substituted for Q in Equation (36).

Products $C_1 A_1$, $C_2 A_2$, $C_3 A_3$, etc., represent the average relative Q_1, Q_2, Q_3, etc., contributing to individual slopes, S_1, S_2, S_3, etc., respectively. Since the relative Q, in the foregoing method for estimating the average slope (see Equation (36)), carries the exponent of $\frac{1}{4}$, its value (or product, CA) may be roughly approximated without affecting the final run-off result materially.

Substitution of $S^{\frac{1}{2}}$ for $S^{\frac{3}{8}}$ in Equation (36) renders a fairly close approximation for the average slope. The resulting slope is a little smaller than that obtained by using $S^{\frac{3}{8}}$.

Average Value of F

Having observed the foregoing method for estimating the average slope, it should be evident that the following equation may be used in arriving at the average value of F, for use with the average slope (substituting $C\,A$ for the relative Q):

$$F^2 = \frac{\dfrac{L_1}{(C_1\,A_1)^{\frac{1}{4}}} + \dfrac{L_2}{(C_2\,A_2)^{\frac{1}{4}}} + \text{etc.}}{\dfrac{L_1}{(C_1\,A_1)^{\frac{1}{4}}\,F_1^2} + \dfrac{L_2}{(C_2\,A_2)^{\frac{1}{4}}\,F_2^2} + \text{etc.}} \qquad \ldots\ldots\ldots\ldots (38)$$

Average Value of C

The proper assignment of values to the run-off coefficient requires an intimate knowledge of all the various factors affecting run-off as well as a fund of experimental information from which to draw. The refinement to which the engineer should resort in establishing its value depends both on the particular problem confronting him and the reliability of available information.

Values for Factor C can be secured only through experimentation by obtaining field measurements for A, i, and Q; and solving by Equation (1). Some of the information presented in current textbooks and past and present publications has been procured on this basis; various tables of values for C (especially for use in storm drain work) are listed. However, much of the available information is of a conflicting nature, due in some cases, no doubt, to unreasonable conceptions, and in others to conclusions drawn from insufficient data concerning the many varying causes of fluctuation.

There is, no doubt, much unpublished information possessed by many of the larger flood-control organizations and others, that is used privately, but which is of potential value to the profession generally. Some of these data are issued from time to time in local publications, while many are retained for aid in the solution of local problems, another portion being still in the experimental stage.

Liberal contributions of this information, coupled with a systematic collection by the Society of existing available data, could be made the first step in a more thorough analysis of Factor C with the view of establishing tentative tables of general value to the profession.

Such tables could not be final; experimentation will be continually needed in the future; but a more concerted effort of the profession in analyzing data on the run-off coefficient would not only be of temporary worth, but its results should point the way to better experiments. To this general fund of information the reader is referred for values of the run-off coefficient. Reference is also made to the conclusions herein under the heading, "Factor C, The Run-off Coefficient".

At present, average values of C for a given water-shed are arrived at by many through comparisons with experimental values obtained on areas of somewhat similar characteristics and size. When a water-shed is considered in parts the average, C, is, of course, the weighted average of separate divi-

sions, for example, $C = \frac{a_1 C_1 + a_2 C_2 \ldots + a_n C_n}{a_1 + a_2 + a_3 \ldots + a_n}$, in which, $a_1, a_2, a_3, \ldots a_n$, represent individual areas composing the water-shed.

When information on the variation of C with time and intensity is possessed, it may be considered advisable to divide the area into approximate time divisions. For areas very irregular in regard to the factors, F and S, the divisions for Factor C may be correlated with those for the other factors.

It should be evident that the increase of the run-off coefficient with the time (as at present practiced by many of the users of the rational method) can be predicated only on the assumption of a water-shed, the surface of which is not in a saturated condition when the maximum run-off producing intensity occurs.

The assumption of a saturated surface condition is, of course, conducive to the maximum run-off. The question naturally arises, when may the surface be considered saturated? Much good information on this point has been given[10] by Ivan E. Houk, M. Am. Soc. C. E. In this connection the chapter of his report entitled, "Rainfall and Run-Off Rates on Saturated Soils",[11] is of particular interest.

It is also worthy of note that proper assumptions on degree of previous saturation pursuant to the use of the rational method cannot be made without due consideration of the manner of making the rainfall curve the use of which is contemplated. The intensity-time-frequency curves in present use (as evidenced by current publications and textbooks) are platted from data selected from portions of storms, in such manner that their use presupposes that precipitation may be expected to occur both before and after the run-off producing intensity represented by a given ordinate.

It seems reasonable, therefore, that in attempting to estimate the probable maximum run-off for the wet season of the year (at which time the storm predicted may be expected to follow closely on the heels of previous rains) through the use of such curves, a saturated surface condition should be assumed for the usual water-shed, the surface of which is composed of ordinary soil, or combinations of soil and impervious materials.

Analysis of Experimental Data.—Equation (39), following, is derived for the purpose of facilitating the analysis of experimental data, leading to the compilation of tables (or diagrams) of Factor C.

For evolving the formula, retention is considered as subdivided into three main parts, namely, evaporation, absorption, and storage.

Evaporation, which herein is meant to include transpiration, is not a great consideration in the process of estimating instantaneous run-off, and usually could be neglected without serious error. The loss to run-off through absorption is large when the soil is dry, and is considerable for saturated soils. Storage, as used in the formula, is intended to represent the loss to run-off of the total precipitation falling in depressions, cavities, etc. Thus, the term includes percolation, when the precipitation is lost in this manner.

[10] "Rainfall and Run-Off in the Miami Valley," by Ivan E. Houk, Technical Repts. Miami Conservancy Dist., Pt. VIII.

[11] *Loc. cit.*, p. 119.

It is significant that the combined rates of evaporation and absorption may be represented in the formula by a single symbol, since they are not altered materially by a change in intensity. The rate of storage, on the other hand, will be greater for an increased intensity. Since all practical areas contain some storage capacity, this fact (in addition to the consideration of evaporation), explains why experimental results never show unity for Factor C, even for a saturated impervious surface.

For a given area, let m = percentage of precipitation, i, that is being stored. Then $m\,i\,A$ = rate of storage, in cubic feet per second, and $i\,A - m\,i\,A$ = rate of run-off, having deducted storage alone, in cublic feet per second. Of the remaining water, evaporation and absorption are taking toll.

Let z = rate of absorption plus rate of evaporation, in cubic feet per second per acre. Then $z\,A - m\,z\,A$ = cubic feet per second being absorbed and evaporated; and, therefore, $(i\,A - m\,i\,A) - (z\,A - m\,z\,A) = i\,A - z\,A + m\,z\,A - m\,i\,A$ = actual run-off, in cubic feet per second.

Substituting this expression for Q in Equation (1), there results: $C\,i\,A = i\,A - z\,A + m\,z\,A - m\,i\,A$, and,

$$C = 1 - \frac{z - m\,z + m\,i}{i} = \left(1 - \frac{z}{i}\right)(1 - m) \ldots\ldots\ldots\ldots (39)$$

In connection with Equation (39), reference should be made to Fig. 6. The curves thereon not only transform the very valuable experiments of Mr. Houk into a form suitable for use in the rational method, but also tend to illustrate the usefulness of Equation (39).

The intercept on the abscissa axis (or the value of intensity when Factor $C = 0$) is represented by the letter, b, in Mr. Houk's report, and by z in Equation (39). There is no letter designation in the Houk report to indicate the percentage of storage, m, of the formula.

Values of m and z for the curves, H', I', J', K', and L', respectively, are shown in Table 7.

TABLE 7.—VALUES OF m AND z FOR CURVES, H', I', J', K', L' AND M' OF FIG. 6

Curves	m	z
H'	0.066	0.20
I'	0.375	0.25
J'	0.048	1.05
K'	0.048	0.87
L'	0.0	0.67
M'	0.0	0.52

Calculations for the values in the line, H', will illustrate. From the straight-line equation, $y = 1.07\,x + 0.20$ (for Line H of the report), derive the value of $\frac{x}{y} = \frac{y - 0.20}{1.07\,y}\left(1 - \frac{0.20}{y}\right)\frac{1}{1.07}$. Since $\frac{x}{y}$ is the equivalent of Factor C of Equation (39), $\frac{1}{1.07} = 1 - m$, and, $m = 0.066$ (see Table 7).

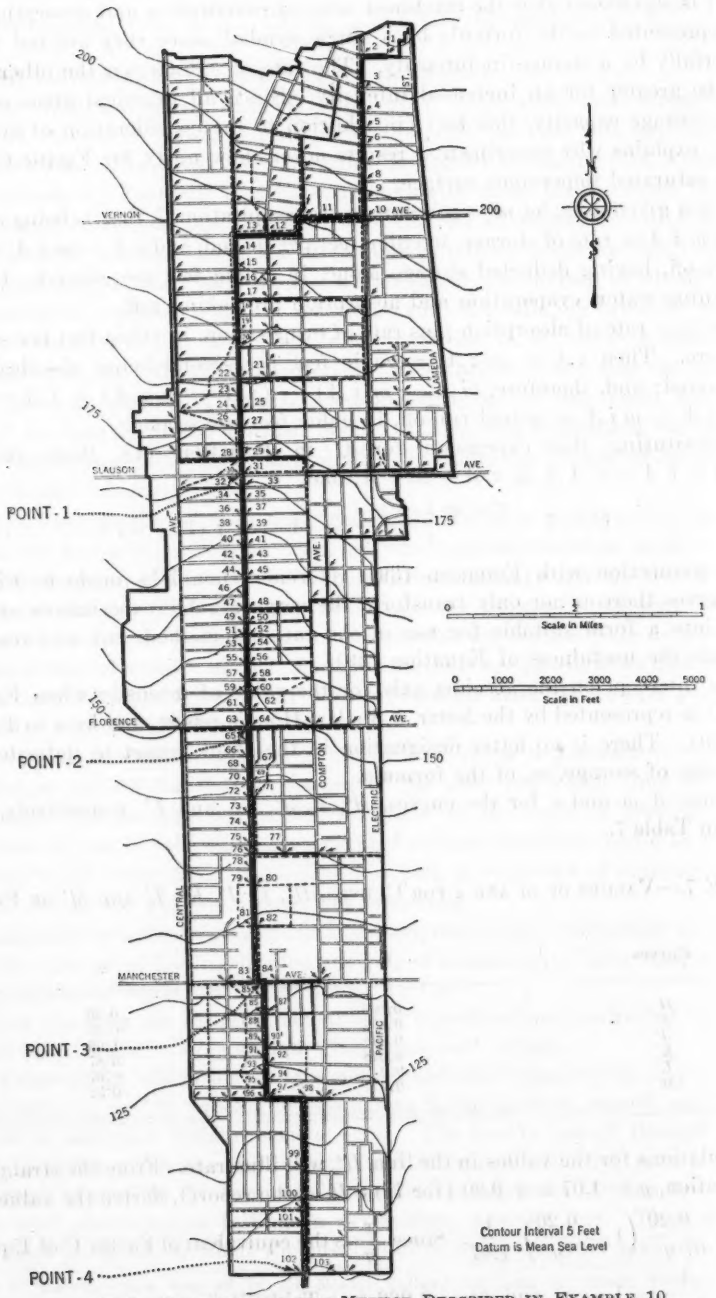


FIG. 6.—ILLUSTRATING THE METHOD DESCRIBED IN EXAMPLE 10

It should be observed (see Table 7), that some storage, as defined in assumptions for the derivation of Equation (39), is recorded (although the areas represented were very small) for all the curves with the exception of L' and M'.

The values of m for Curves H', J', K', L', and M' are small because the soil of the areas experimented upon was packed. It seems that the extremely high value for m of Curve I' was caused by spading the soil, thus permitting a certain percentage of the precipitation (regardless of its intensity) to enter cavities directly. In such case, the water due to a certain intensity and time of sprinkling would merely fill the cavities. Additions above this quantity would still be lost because of the nature of the subsoil and the head developed in the cavities.

Suggestions for Use of Formulas

Method No. 1.—The first method involves seven distinct steps, as follows:

1.—Decide on the type of rainfall curve, and reduce Equation (24) accordingly:

(*a*) In the absence of rainfall data and for approximate estimates, use Equation (28) or Equation (29).
(*b*) Establish H.
(*c*) Determine R_H.

2.—Determine A and locate the main channel of flow (if necessary do this by trial, using approximate values for nucleus, C, F, S, P, etc.). Establish L, and $\frac{L}{W}$.

3.—Determine average values for Factors C, F, and S, for the water-shed, being guided by admonitions in the preceding parts of the paper and judgment concerning the requirements of the particular problem.

4.—Determine C', A', L', $\frac{L'}{W'}$, and $\frac{A}{A'}$; assign a time, t', to A', and calculate Q' from $Q' = C'\, i'\, A'$. (When the use of Table 3 is contemplated, or when the assumption is that the storm is of uniform intensity for the time of concentration of the water-shed, i' should be the approximate intensity of the water-shed, and t' should be based on this intensity.

When the use of Table 4 is contemplated, or when it is desired to take advantage of the factor of safety rendered by the use of a rainfall curve of the type of Equation (3) in the manner ordinarily employed in the detail process of the rational method, $i' = \frac{K}{(t')^{0.5}} = R_H \frac{(60\ H)^{0.5}}{(t')^{0.5}}$ (for t', in minutes).

5.—Obtain $v' = \frac{L'}{t'}$, for t', in seconds, and $V' =$ velocity of Q', based on average values of F and S for the water-shed, $= \frac{F^2}{S^{\frac{1}{8}}} s^{\frac{1}{2}} (Q')^{\frac{1}{4}}$. Determine $\frac{v'}{V'}$ and correct this value of P_1 in Fig. 3, giving P'.

6.—With the corrected value of P' enter Table 3, or Table 4, in the proper columns for $\frac{L}{W}$ and $\frac{A}{A'}$, being guided by Fig. 2 and by guide lines for the manner of concentration in Fig. 4, finding P.

7.—Substitute the values of Factors C, A, R_H, S, P, F, and L in the reduced formula and solve for Q. It should be remembered that:

(*a*) Length L includes L' of the nucleus.
(*b*) The average, C, includes consideration of C' of the nucleus.
(*c*) The average slope for use in formulas is the average slope of the main channel of flow, exclusive of the nucleus.
(*d*) The average, F, is the average value of F for the main channel of flow, exclusive of the nucleus.

This method offers opportunity for varying the effect of the inlet area, or nucleus, in much the same manner as may be followed in the detail process of the rational method. As has been shown, the assignment of a value for P' on a given nucleus is the equivalent of allowing so much inlet, or initial, time. The method will be found valuable, depending on the water-shed and the purpose in view. For instance, it may be considered advisable to allow a certain lapse of time to prepare a regular water-shed to begin shedding the water by allowing for storage in small irregularities of surface, leaves, vegetation, etc. (this, however, is a special consideration), or, again, it may be of advantage to use Method No. 1, when the types and slopes of channels through which the inlet areas contribute, differ radically from that of the main channel of flow.

Method No. 2.—It should be borne in mind, however, that Method No. 1, is not the only manner in which the formulas may be found useful. Broadly speaking, they may be used in the same way as the ordinary empirical formulas, with the added advantage of permitting the engineer to allow for variables, which the use of existing formulas prohibits (see the heading, "Inadequacy of Empirical Formulas").

When used in this manner the length, L, should be the full distance from the most remote part of the water-shed to its outlet, and Factors S and F should represent average values for the full length.

Furthermore, Factor P should be chosen from the column of Table 3 headed by $P' = 0.5$, and under the section where $\frac{A}{A'} = 262\ 144$ (approaching infinity, see the statement concluding the heading, "Method of Arriving at Values of Factor P"). The value of $\frac{L}{W}$ for the water-shed, and the manner of concentration, should be considered, of course, in arriving at the value of P.

This method is short, but, if proper judgment is exercised in selecting values of the various factors of the formula used, it affords a means of arriving at a quick approximate Q and, consequently, the time of concentration of a given water-shed for the assumptions. Having computed Q, the intensity is found from Equation (1), and the time of concentration is determined by referring the intensity to the rainfall curve on which the formula used is based.

Indeed, this method may be used in all ordinary cases without recourse to Method No. 1, admitting a small error, which is less on the average than the automatic error of the detail process mentioned under the heading, "Use of Rainfall Curve" (see, also, "Examples:" "Remarks on Example 10").

After the time has been found, simply add or deduct the difference in inlet time, referring the new time to the rainfall curve for the intensity to be used in Equation (1).

It should be evident that the first solution by this method entails the assumption that the water is ready to begin concentrating immediately, the time of the inlet area, or nucleus, being contained in the total time of the water-shed as rendered by average values of the various factors of the formula for the entire area. Any increase (or decrease) in the inlet time above (or below) this assumption may be applied as suggested herein.

Special Case.—Attention should be called to a special case sometimes encountered in the rational method of computing run-off. In mountainous territory it is often found that the higher altitudes of a given water-shed may reasonably be expected to receive a greater intensity for an assumed period of frequency than the lower portions. In attacking the solution of this problem the territory may be divided into zones, to each of which is assigned the probable R_H that may be expected to occur. Thus, where R_1 represents the probable average intensity over any one zone throughout a period of 1 hour (for a water-shed divided into three parts), the zone of highest altitude may be assigned a value, $R_1 = 3$ in., the middle zone, $R_1 = 2.50$ in., and the lowermost zone, $R_1 = 2$ in. per hour.

In the detail process of the rational method the problem is sometimes approached as follows: On the diagram of rainfall curves, that curve representing the assigned intensity requirements (such as $R_1 = 3$ in.), is chosen; this curve is followed until all the individual areas of the first zone have contributed; then transfer is made to the required curve for the next zone, such as $R_1 = 2.50$ in. per hour (without making reduction at the point of transition), the second curve being used as the first. In this manner the entire water-shed is covered. The method is an approach to a reasonable solution for the conditions assumed. The method results in the use of an average rainfall curve for the several zones.

In using the formulas and methods herein outlined the average value of R_H may be obtained for the water-shed through weighting the individual zone values of R_H by the respective zone areas. The average value of R_H is to be used, of course, with the rainfall curve to which it belongs.

Other Coefficients.—It is unfortunate that more information is not available concerning the distribution of rainfall especially over the larger areas. It is possible that more accurate information on this point could be utilized through three possibilities, as follows:

(*a*) Alteration of the frequency periods (or values for a given frequency) of the time-intensity-frequency curves in present use.

(*b*) Coefficients of distribution.
(*c*) The use of only a part of a given drainage area instead of the whole.

The writers believe that it is customary at present among American engineers to use rainfall curves as if the intensity were uniform over the entire area throughout the time of concentration. This practice (in the absence of more thorough complete information on the distribution of rainfall) is based, no doubt, on the common observation that rains of lighter intensity and longer duration (required for consideration of the larger areas) are likely to be steadier and of wider distribution than those rendering higher rates of precipitation, which are more generally applicable to the smaller areas.

The time-intensity-frequency curves in present use are usually developed from the records of individual gauges. In this case it would be unwise to assume that the intensity designated by a certain ordinate on a curve represents the rate of precipitation at the eye or focus of the storm; or that it should be used to indicate the maximum rate within an area on which the intensity is unequally distributed at a given moment.

Coefficients based on this assumption, therefore, should not be used as direct multiple factors in reducing the run-off results that may be obtained through the use of the curves developed from the records of individual gauges.

As the writers view the problem the following procedure (in the light of present and future information) should be followed:

1.—Efforts should be made to ascertain whether or not the intensities rendered by curves in present use may reasonably be applied over the entire area.
2.—If such intensities can not be applied over the entire area, then endeavor should be made to apply the information through the foregoing suggested possibilities, (*a*), (*b*), or (*c*).

The "coefficient of retention" is accounted for in the run-off coefficient as used by American engineers. (See the heading, "Factor *C*, The Run-off Coefficient".)

The "coefficient of retardation" is defined as the ratio of the contributing area to the total drainage area when the maximum discharge is caused by a storm which lasts for a lesser length of time than the time of concentration of the total drainage area. This possibility should be taken into account, especially for some areas, very irregular in regard to shape and physical characteristics.

Instead of using a coefficient, however, the problem should be solved by trial, use being made of that portion of the area rendering the maximum. The formulas herein (usually through Method No. 2), may be used advantageously for this purpose. Other writers have discussed this phase of the rational method.

Remarks.—The outstanding feature of the rational method is its conception of the time element in the estimation of maximum storm flows, affording, as it does, through the medium of its assumptions, a reasonable basis for predicting the probable intensities and frequencies of storms. It should be emphasized, however, that the solution of the run-off problem, although

approached more easily through the special assumptions of the rational method, is still indeterminate. This statement may be understood more easily when it is recognized that there is no conceivable way of arriving at a perfect solution under the assumptions, because it is necessary to approach accuracy by trial.

The detail method is only an approach. For instance, at the beginning of the process, it is necessary to assign a time for the concentration of the initial area. This time, among other considerations, depends on the intensity. Since the assumption is that for a given water-shed the intensity shall be uniform over the entire area throughout its time of concentration, and, since its value cannot be known in advance, the assignment of the inlet time may be made only as an approach to accuracy under the assumption. Likewise, in the assignment of the run-off coefficient both to the initial area and the remainder of the water-shed (since its value depends on the intensity), accuracy can only be approached by trial. Therefore, it has not been the common practice of users of the detail rational method to consider the effect of intensity on either the inlet time or the value of the run-off coefficient.

Intelligent use of the formulas, requires not only a familiarity with the foregoing derivations and assumptions, but also a broad understanding of the run-off problem. The foregoing work, it is hoped, may serve as an aid in the solution of a difficult (and somewhat neglected) problem.

Examples

The following examples are intended to illustrate the application of the formulas.

Example 6.—Assume that for a given locality the intensity curve of the type, $i = \frac{K}{t^{0.5}}$ (Equation (3)), is considered adaptable, that, $R_H = R_1$, and that the time period, H, is 1 hour. For this case, Equation (29) may be used. Let $R_1 = 2.0 =$ the expected rainfall intensity, in inches per hour, for a 1-hour period.

The study of a given area reveals that the water-shed area is somewhat similar in shape and manner of concentration to Type 18, Fig. 2. The dimensional data are: $A = 1\,294$ acres; $L = 20\,328$ ft.; $S = 10$ ft. per 1 000 ft.; and $C = 0.3$.

The main channel of flow is assumed to be a pipe with Kutter's $n = 0.013$. Hence, from Table 11 (Appendix II), $F = 6.63$.

Examination of the nucleus (or initial area) reveals that: $A' = 5$ acres; $L' = 700$ ft.; and $C' = 0.2$.

The following factors are estimated: $\frac{A}{A'} = 259$; $\frac{L}{W} = \frac{(20\,328)^2}{43\,560 \times 1\,294} = 7.3$; $\frac{L'}{W'} = \frac{(700)^2}{43\,560 \times 5} = 2\,25$.

If the use of Table 4 is contemplated (or if it is desired to take advantage of the small factor of safety rendered by the use of a rainfall curve, Equation (3), as ordinarily followed in the detail process of the rational method), and

a study of the nucleus calls for an assigned time, $t' = 10$ min., for this assumption; then,

$$Q' = C'\, i'\, A' = C'\, A' \frac{7.75\, R_H \sqrt{H}}{\sqrt{t'}} = \frac{0.2 \times 7.75 \times 2.0}{\sqrt{10}} \times 5.0$$
$$= 4.9 \text{ cu. ft. per sec.}$$

If that type of storm based on a uniform, or constant, intensity for the time of concentration had been assumed, it would have been necessary to base the assignment of t' on this assumption, using the approximate intensity for the water-shed; also, $Q' = C'i'A'$, wherein i' is the approximate intensity of the water-shed.

For the velocity ratio:

$$v' = \frac{L'}{t'} = \frac{700}{10 \times 60} = 1.17 \text{ ft. per sec.}$$

$$V' = \frac{F^2}{S^{\frac{1}{3}}} \sqrt{s}\ (Q')^{\frac{1}{4}} = \frac{(6.63)^2}{(10)^{\frac{1}{3}}} \times \sqrt{0.01}\ (4.9)^{\frac{1}{4}} = 4.87 \text{ ft. per sec.}$$

and,

$$\frac{v'}{V'} = \frac{1.17}{4.87} = 0.24 = P_1$$

Entering Fig. 3 with $\frac{L'}{W'} = 2.25$, and $P_1 = 0.24$,

$$P' = 1.24 \times P_1 = 1.24 \times 0.24 = 0.3$$

Entering Table 4, with $P' = 0.3$, $\frac{A}{A'} = 259$, and $\frac{L}{W} = 7.3$, and guided by Figs. 2 and 4 to medium concentration for area, Type 18, $P = 0.51$. Hence,

$$\sqrt{\frac{P}{L}} = \sqrt{\frac{0.51}{20\,328}} = 0.005 = B.$$

Substituting values for the various factors in Equation (29),

$$Q = 15\ (0.3 \times 1\,294 \times 2.0 \times 6.63 \times 0.005)^{1.1429}\ (10)^{0.2143}$$
$$= 1\,006 \text{ cu. ft. per sec.}$$

The intensity and time of concentration for the foregoing assumptions may be found from Equation (1), thus, $i = \frac{1\,006}{1\,294 \times 0.3} = 2.59$, and from Equation (26), $t = \frac{60\ (2.0)^2}{(2.59)^2} = 35.7$ min.

Example 7.—Assume that information on rainfall and the consideration of larger areas point to the use of a longer time period, or $H = 6$ hours; then, $R_H = R_6$. Again, assume that the rainfall curve, Equation (3), is adaptable. In this case Equation (28) may be used, or, substituting $H = 6$,

$$Q = 41.7\ (C\ A\ R_6\ F\ B)^{1.1429}\ S^{0.2143} \ldots\ldots\ldots\ldots\ldots\ (40)$$

In $R_H = \frac{X}{H}$, or, $R_6 = \frac{X}{6}$, assume that $X = 4.8 =$ the expected inches of rain for a period of 6 hours; then, $R_6 = \frac{4.8}{6} = 0.8$ in. per hour.

From a study of a given area it develops that the shape and manner of concentration are approximately midway between Types 16 and 17, Fig. 2. Furthermore, $A = 500$ sq. miles $= 320\,000$ acres; $A' = 8$ acres; $L = 223\,600$ ft.; $L' = 1\,200$ ft.; $S = 5$ ft. per 1 000 ft.; $C = 0.2$; $C' = 0.25$; and $F = 5.7$. By estimate, $\frac{A}{A'} = \frac{320\,000}{8} = 40\,000$; $\frac{L}{W} = \frac{(223\,600)^2}{43\,560 \times 320\,000} = 3.6$; and $\frac{L'}{W'} = \frac{(1\,200)^2}{43\,560 \times 8} = 4.0$.

The use of Table 3 is contemplated. Assign an approximate average intensity for the water-shed, of 0.75 in. per hour. Examination of the nucleus points to the assignment of 15 min. for its time of concentration. Then, following Equation (1),

$$Q' = C'\, i'\, A' = 1.5 \text{ cu. ft. per sec.}$$

$$v' = \frac{L'}{t'} = 1.33 \text{ ft. per sec.}$$

$$V' = \frac{F^2}{S^{\frac{1}{8}}} \sqrt{s}\, (Q')^{\frac{1}{4}} = 2.08 \text{ ft. per sec.}$$

and,

$$\frac{v'}{V'} = 0.64 = P_1 = P'$$

(no correction is necessary since $\frac{L'}{W'}$ of the nucleus is approximately 4).

Entering Table 3 with $P' = 0.64$, $\frac{A}{A'} = 40\,000$; and $\frac{L}{W} = 3.6$, guided by Figs. 2 and 4, to approximately midway between medium and delayed concentration, P is found equal to 0.48 and $\sqrt{\frac{P}{L}} = 0.001465 = B$. Substituting values for the various factors in Equation (40),

$$Q = 41.7\ (427.5)^{1.1429}\ (5)^{0.2143} = 59\,740 \text{ cu. ft. per sec.}$$

The intensity by Equation (1) and the time of concentration by Equation (26) are, $i = 0.933$, and $t = 4.4$ hours, respectively.

Example 8.—Let the rainfall curve for a given locality be represented by $i = \frac{K}{t^{0.6}}$. Assume that $R_H = R_1$, or that $H =$ a 1-hour period. For this purpose Equation (24) reduces to:

$$Q = 28.3 \left(\frac{P}{L}\right)^{0.7059} (C\, A\, R_1)^{1.1765}\, F^{1.4118}\, S^{0.2647} \ldots\ldots\ldots (41)$$

Now, assume that the values for the various factors are the same as those used in Example 6; then, by Equation (41) $Q = 1\,069$ cu. ft. per sec.

Example 9.—Let the rainfall curve be the same as in Example 8 and let $R_H = R_6$ and $H = 6$; then, Equation (24) reduces to:

$$Q = 100.2 \left(\frac{P}{L}\right)^{0.7059} (C\, A\, R_6)^{1.1765}\, F^{1.4118}\, S^{0.2647} \ldots\ldots\ldots (42)$$

Substituting in Equation (42) the same factors used in Example 7, $Q = 62\,003$ cu. ft. per sec.

Remarks Concerning Examples 6, 7, 8, and 9.—The run-off, Q (= 1 069), for Example 8, may be approximated by referring the time, 35.7 min. (solved by the use of Equation (29)), to the rainfall curve,

$$i = \frac{K}{t^{0.6}} \quad \ldots\ldots (43)$$

thus obtaining a new intensity for use in Equation (1).

In the absence of the curve, Equation (43), the intensity may be calculated directly from Equation (43) by substituting for K its value when $H = 1$ hour, and $R_H = R_1$; thus,

$$i = \frac{(60)^{0.6}}{t^{0.6}} \quad \ldots\ldots (44)$$

This results in the value, $i = \frac{23.33}{35.7^{0.6}} = 2.73$. By Equation (1), the corrected run-off equals $0.3 \times 2.73 \times 1\,294 = 1\,060$ cu. ft. per sec. The percentage of accuracy is expressed by $\frac{1\,060}{1\,069} = 99.2$.

Likewise, the run-off, $Q = 62\,003$, for Example 9 may be approximated by referring the time, 264 min. (solved by the use of Equation (28)), to the curve of Equation (43) with the result that $i = 0.964$.

By Equation (1) the corrected run-off,

$$Q = 0.2 \times 0.964 \times 320\,000 = 61\,696 \text{ cu. ft. per sec.}$$

In this case the percentage of accuracy is $\frac{61\,696}{62\,003} = 99.6$.

The result to be expected from a given problem will not always be as close as the foregoing percentages indicate, since the shape, size, and manner of concentration of the area, will have some effect. However, when the adopted rainfall curve does not vary widely, the foregoing method may be used to give a quick approximate result, accurate enough for most practical purposes, for variations in the value of exponent, e, Equation (2), as encountered in common practice.

Similar transitions may be made, of course, to curves of other types, such as,

$$i = \frac{a}{b + t} \quad \ldots\ldots (45)$$

Example 10.—This example is shown for the purposes of illustrating the hydraulic construction of the formulas and making certain comparisons with the usual detail method.

Fig. 7 was selected from a topographical map and represents a small water-shed (rather thoroughly divided into individual areas by streets), with contour intervals of 5 ft. For comparative purposes it is necessary to place

the tabling process and the formulas on the same basis, hence, the following assumptions are made arbitrarily:

(*a*) The rainfall curve is of the type of Equation (3), with $R_H = R_1 = 1.25$ in. per hour for a 1-hour period. Since $R_1 = \frac{K}{(60)^{0.5}}$,

$$K = (60)^{0.5} \times 1.25 = 9.69$$

or,

$$i = \frac{9.69}{t^{0.5}} \qquad (46)$$

(*b*) Circular pipe (Kutter's $n = 0.013$) is used throughout the main, or time-giving, channel of flow.

(*c*) The run-off coefficient, C, is equal to 0.4 throughout.

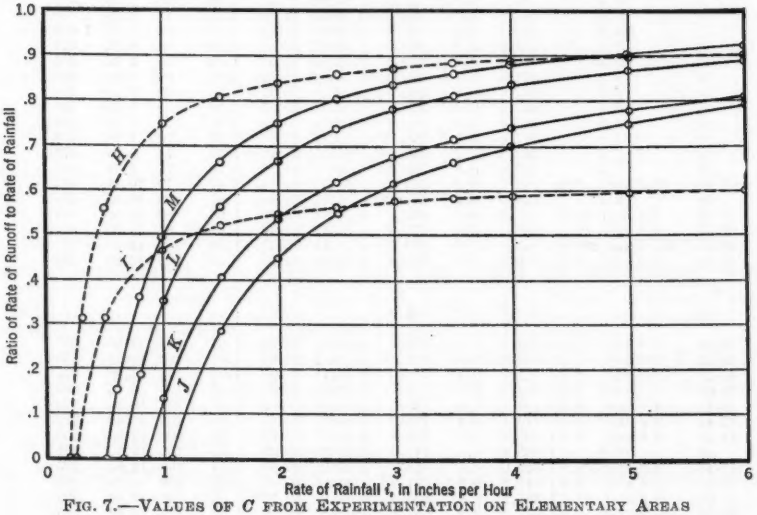


FIG. 7.—VALUES OF C FROM EXPERIMENTATION ON ELEMENTARY AREAS

Area No. 1 of Fig. 7 is chosen as the initial area for the tabling process shown in condensed form in Table 8. It is further assumed that the inlet Area No. 1 (a small water-shed) concentrates in the manner shown for Type 3, Fig. 2. For the initial area: $A = 8$ acres; $L = 1\,200$ ft.; $S = 3$ (for a fall of 3.6 ft. in 1 200 ft.); $C = 0.4$; $R_1 = 1.25$; and P (from Table 4, making use of "Suggestions for Use of Formulas, Method No. 2,") $= 0.53$. Using Equation (29) (due to the foregoing assumption for the rainfall curve: $B = \sqrt{\frac{P}{L}} = 0.021$, and $Q = 15\ (0.4 \times 8 \times 1.25 \times 6.63 \times 0.021)^{1.1429}\ (3)^{0.2143} = 9.8$ cu. ft. per sec. By Equation (1) then, $i = 3.06$, and by Equation (26),

$$t = \frac{60 \times 1.25^{-2}}{3.06^2} = 10 \text{ min.}$$

For the initial area, 10 min. represents a value that would be found for its time of concentration (on the foregoing assumptions) if it were tabled down from an infinitesimal beginning by the detail process. That value is used to

TABLE 8.—CONDENSED ILLUSTRATION OF THE TABLING PROCESS TO ACCOMPANY EXAMPLE 10

Water-Shed No.	Area drained	Individual area, in acres	Total area, in acres, (A)	Coefficient of run-off (C)	Run-off, $Q = CiA$	Grade of drain (s)	Time, in minutes (t)	Intensity, $i = \frac{9.69}{t^{0.5}}$	Size of pipe, in inches	Velocity, in feet per second	Length, in feet	Total length, in feet†	Total fall, in feet†
	1	8.0	8.0	0.4	9.8	0.00395	10.00	3.07	21	4.07	530		
	2	6.7	14.7	0.4	16.4	0.00395	12.17	2.78	26	4.76	700		
	3	16.5	31.2	0.4	31.6	0.00395	14.62	2.53	33	5.64	490		
	4	14.0	45.2	0.4	43.8	0.00468	16.07	2.42	36	6.52	360		
	5	8.1	53.3	0.4	50.1	0.00468	16.99	2.35	38	6.78	330		
	6	8.4	61.7	0.4	56.8	0.00468	17.80	2.30	39	6.90	375		
	7	8.8	70.5	0.4	63.2	0.00468	18.71	2.24	41	7.14	375		
	8	9.8	80.3	0.4	70.3	0.00468	19.58	2.19	43	7.38	375		
	9	10.5	90.8	0.4	77.7	0.00468	20.43	2.14	44	7.50	385		
	10	10.7	101.5	0.4	85.3	0.00331	21.28	2.10	49	6.79	1 300		
	11	108.6	210.1	0.4	164.6	0.00331	24.48	1.96	62	7.97	1 000		
	12	41.1	251.2	0.4	188.9	0.00331	26.57	1.88	65	8.22	675		
	13	75.2	326.4	0.4	239.0	0.00411	27.94	1.83	69	9.53	335		
	14	5.1	331.5	0.4	240.0	0.00411	28.52	1.81	69	9.53	335		
	15	5.5	337.0	0.4	242.6	0.00411	29.11	1.80	69	9.53	320		
	16	3.5	340.5	0.4	242.4	0.00411	29.67	1.78	69	9.53	320		
	17	75.3	315.8	0.4	292.7	0.00411	30.23	1.76	74	9.98	325		
	18	1.8	417.6	0.4	292.3*	0.00411	30.77	1.75	74	9.98	320		
	19	1.5	419.1	0.4	289.9*	0.00411	31.30	1.73	74	9.98	275		
	20	1.5	420.6	0.4	289.3*	0.00411	31.76	1.72	74	9.98	685		
	21-22	93.9 and 5.2	519.7	0.4	351.4	0.00411	32.91	1.69	79	10.42	355		
	23	4.9	524.6	0.4	350.4*	0.00411	33.48	1.67	79	10.42	320		
	24-25	7.2 and 19.7	551.5	0.4	366.2	0.00411	33.99	1.66	80	10.50	345		
	26-27	4.8 and 10.1	566.4	0.4	373.9	0.00411	34.44	1.65	81	10.59	660		
	28-29	232.7 and 260.0	1 060.0	0.4	691.9	0.00411	35.47	1.63	102	12.30	335		
	30-31	1.6 and 9.2	1 070.8	0.4	693.8	0.00411	35.93	1.62	102	12.30	270		
1.	1 to 31	.....	1 070.8	0 4	693.8	.....	35.93	....	....	.....	.....	11 825	47.19
	32-33	5.6 and 8.0	1 084.4	0.4	698.4	0.00411	36.29	1.61	102	12.30	265		
	34-35	8.2 and 7.8	1 100.4	6.4	704.3	0.00411	36.65	1.60	103	12.38	295		
	36-37	8.6 and 8.9	1 117.9	0.4	711.0	0.00411	37.05	1.59	103	12.38	325		
	38-39	9.8 and 9.5	1 137.2	6.4	718.7	0.00411	37.49	1.58	104	12.46	320		
	40-41	9.8 and 9.5	1 156.5	0.4	726.3	0.00411	37.92	1.57	104	12.46	325		
	42-43	9.7 and 9.9	1 176.1	0.4	733.8	0.00411	38.35	1.56	104	12.46	320		
	44-45	9.6 and 9.5	1 195.2	0.4	745.8	0.00411	38.78	1.56	105	12.53	350		
	46	10.6	1 205.8	0.4	747.6	0.00411	39.25	1.55	105	12.53	335		
	47-48	9.2 and 114.0	1 329.0	0.4	818.7	0.00411	39.69	1.54	109	12.84	295		
	49-50	8.4 and 4.4	1 341.8	0.4	821.2	0.00411	40.07	1.53	109	12.84	280		
	51-52	8.3 and 4.1	1 354.2	0.4	823.4	0.00411	40.44	1.52	109	12.84	320		
	53-54	8.5 and 4.3	1 367.0	0.4	831.1	0.00411	40.85	1.52	109	12.84	285		
	55-56	8.6 and 22.0	1 397.6	0.4	844.1	0.00368	41.22	1.51	112	12.36	295		
	57-58	9.2 and 8.8	1 415.6	0.4	849.3	0.00368	41.62	1.50	113	12.42	290		
	59-60	8.5 and 6.9	1 431.0	0.4	858.6	0.00368	42.01	1.50	113	12.42	360		
	61-62	10.6 and 5.7	1 447.3	0.4	862.6	0.00368	42.49	1.49	113	12.42	360		
	63	109.4	.....	....	.....	.....	....	....	....	.....	.....		
	64	96.3	1 653.0	0.4	978.6	0.00324	42.98	1.48	122	12.08	295		
2.	1 to 64	.....	1 653.0	0.4	978.6	.....	42.98	....	....	.....	.....	17 115	68.37
	65	9.8	1 662.8	0.4	977.7*	0.00324	43.38	1.47	122	12.07	290		
	66	8.8	1 671.6	0.4	976.2*	0.00324	43.78	1.46	122	12.06	310		
	67-68	8.5 and 8.7	1 688.8	0.4	986.2	0.00324	44.21	1.46	123	12.09	290		
	69-70	2.2 and 8.6	1 699.6	0.4	985.7*	0.00324	44.61	1.45	123	12.09	300		
	71-72	1.4 and 8.8	1 709.8	0.4	984.8*	0.00324	45.02	1.44	123	12.09	320		
	73	9.2	1 719.0	0.4	990.1	0.00324	45.47	1.44	123	12.09	330		
	74	9.2	1 728.2	0.4	988.6*	0.00324	45.92	1.43	123	12.09	340		
	75	9.2	1 737.4	0.4	986.9*	0.00324	46.39	1.42	123	12.09	210		
	76-77	4.9 and 145.5	1 887.8	0.4	1 072.2	0.00324	46.68	1.42	126	12.35	325		
	78	6.7	1 894.5	0.4	1 068.5*	0.00324	47.12	1.41	126	12.33	325		
	79-80	4.7 and 20.0	1 919.2	0.4	1 082.5	0.00324	47.56	1.41	127	12.36	850		
	81-82	5.9 and 14.8	1 939.9	0.4	1 078.6*	0.00324	48.70	1.39	127	12.35	1 175		
	83	61.1	2 001.0	0.4	1 096.5	0.00280	50.29	1.37	131	11.78	200		
	84	125.3	2 126.3	0.4	1 156.7	0.00280	50.57	1.36	134	11.90	330		
3.	1 to 84	.....	2 126.3	0.4	1 156.7	.....	50.57	....	....	.....	.....	22 675	86.29

* The greatest Q above was used.

† Excluding initial area.

TABLE 8.—(*Continued.*)

Water-Shed No.	Area drained	Individual area, in acres	Total area, in acres (A)	Coefficient of run-off (C)	Run-off, $Q = CiA$	Grade of drain (s)	Time, in minutes (t)	Intensity, $i = \frac{9.69}{t^{0.5}}$	Size of pipe, in inches	Velocity, in feet per second	Length, in feet	Total length, in feet†	Total fall, in feet†
	85	4.5	2 130.8	0.4	1 159.1	0.00280	51.03	1.36	134	11.90	320		
	86–87	2.6 and 4.0	2 137.4	0.4	1 154.3*	0.00280	51.48	1.35	134	11.90	330		
	88	2.7	2 140.1	0.4	1 151.3*	0.00280	51.94	1.35	134	11.90	330		
	89	2.7	2 142.8	0.4	1 148.5*	0.00280	52.41	1.34	134	11.90	90		
	90	29.0	2 171.8	0.4	1 161.5	0.00280	52.53	1.34	135	11.95	210		
	91	2.7	2 174.5	0.4	1 159.4*	0.00280	52.82	1.33	135	11.95	120		
	92	6.5	2 181.0	0.4	1 161.2*	0.00280	52.99	1.33	135	11.95	195		
	93	3.3	2 184.3	0.4	1 160.3*	0.00280	53.26	1.33	135	11.95	165		
	94	7.1	2 191.4	0.4	1 161.5	0.00280	53.49	1.33	135	11.95	150		
	95	3.0	2 194.4	0.4	1 160.5*	0.00280	53.70	1.32	135	11.95	235		
	96–97	41.2 and 3.5	2 239.1	0.4	1 180.4	0.00231	54.03	1.32	140	11.13	775		
	98	85.2	2 324.3	0.4	1 212.3	0.00231	55.19	1.30	142	11.20	1 480		
	99	83.3	2 407.6	0.4	1 231.7	0.00338	57.39	1.28	132	12.95	870		
	100	28.6	2 436.2	0.4	1 234.7	0.00338	58.51	1.27	133	12.95	325		
	101	10.1	2 446.3	0.4	1 234.9	0.00338	58.93	1.26	133	12.95	1 025		
	102–103	35.1 and 151.6	2 633.0	0.4	1 314.4	0.00515	60.25	1.25	126	15.45	1 300		
4.	1 to 103	.....	2 633.0	0.4	1 314.4	.....	60.25	....	....	.....	.....	29 625	105.22

* The greatest Q above was used.
† Excluding initial area.

start the tabling process of Table 8, using Fig. 7. The four water-sheds selected are numbered 1, 2, 3, and 4 on Table 8 and are represented, respectively, by the total areas above Points 1, 2, 3, and 4, of Fig. 7. The velocities in Table 8 were calculated by the use of Kutter's formula.

Comparison of Times of Concentration and Run-off by Different Methods.—The time and run-off, determined by five processes, are arranged for comparison in Table 9. The explanation of this table may be itemized in the form of eight remarks, as follows:

(1) Values in Columns (2) and (3) of Table 9 were selected from Table 8.

(2) Those in Columns (4) and (5) were calculated by the use of Equation (29). The computations for Water-Shed No. 2 (Fig. 7) will illustrate the process.

The following were selected from Table 8: $A = 1\,653$ acres; $L = 17\,115 + 1\,200$ (length in initial area $= 1\,200$ ft.) $= 18\,315$ ft.; $S = \frac{\text{Total fall}}{\text{Total length}} = \frac{68.40 + 3.60}{18\,315} = 3.93$ ft. per 1 000 ft.; $P = \frac{v}{V}$; $v = \frac{L}{t} = \frac{18\,315}{60 \times 42.98} = 7.1$; V (using Kutter's formula, $n = 0.013$, for circular pipe, $S = 3.93$, and $Q = 978.5$) $= 13.1$; and $P = \frac{7.1}{13.1} = 0.54$; and $\sqrt{\frac{P}{L}} = 0.00543 = B$. Solving,

$$Q = 15\,(0.4 \times 1.653 \times 1.25 \times 6.63 \times 0.00543)^{1.1429}\,(3.93)^{0.2143}$$
$$= 971.5 \text{ cu. ft. per sec.}$$

Again, substituting correct values in Equations (1) and (26), respectively, $i = 1.469$ and $t = 43.4$ min.

(3) Columns (6) and (7), Table 9 were calculated by Equation (29) using Table 3, and following Method No. 2. All factors for use in Equation

TABLE 9.—COMPARISON OF TIMES OF CONCENTRATION AND RUN-OFF FOR VARIOUS PROCESSES OF DETERMINATION

Water-Shed No. (see Fig. 6)	From Table 8, Using 10 Minutes Initial Time		Formula, Using Table 4, Method 2		Formula, Using Table 3, Method 2		Ratio	Tabling, Using 20 Minutes Initial Time		Formula, Using Table 4, Method 1		Ratio
	Time t_a, in seconds	Run-off, Q_a, in cubic feet per second	t_b	Q_b	t_c	Q_c	$\frac{Q_b}{Q_c}$	t_1	Q_1	t_2	Q_2	$\frac{\sqrt{t_1}}{\sqrt{t_0+10}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
1	35.9	694	35.9	692	41.3	646	1.07	47.6	600	48.0	593	1.02
2	43.0	979	43.4	972	51.0	897	1.08	54.9	866	55.8	857	1.02
3	50.6	1 157	51.3	1 149	60.8	1 056	1.09	62.5	1 048	63.3	1 035	1.02
4	60.3	1 314	60.5	1 311	72.0	1 202	1.09	72.4	1 200	73.4	1 190	1.02

(29) are the same as those used in the computations for Columns (4) and (5) of Table 9, with the exception of Factor P. This factor was selected from Table 3, corresponding in position to the values chosen from Table 4 for Columns (4) and (5) of Table 9.

(4) Columns (9) and (10) were obtained by "tabling" Fig. 7 in detail, with the same assumptions and data as in Table 8, except that 20 min. were used for the time of the initial area. The table is not reproduced in this paper.

(5) Columns (11) and (12) were calculated by use of Equation (29) and Table 4 and by following Method No. 1, assuming a 20-min. inlet time on the initial area.

(6) It may be noted that Q_a and Q_b, as well as Q_1 and Q_2, in Table 9, are practically equal for a given water-shed. This fact indicates the correctness of the construction of Equation (29) which was reduced from Equation (24). Since the detail process and the equation were placed on the same basis in each case, it shows Methods Nos. 1 and 2 to be logical in approach for the assumptions, and serves to testify to the accuracy of the particular value of 6.63 for pipe with Kutter's $n = 0.013$, and (to a certain extent) the construction of Table 4. The construction of Table 9 was not, however, mainly for the purpose of proving the Points 1 to 5. The logic in the derivation of the formulas is contained in this paper under the proper headings, and the methods of arriving at Factors P and F are fully explained.

Values Q_a, Q_b, Q_1, and Q_2 contain the error mentioned herein under the heading "Use of Rainfall Curve". Values Q_c do not contain this error, being evolved through the use of Table 3 with the formula. Values of the ratio, $\frac{Q_b}{Q_c}$ show the possible magnitude of the error rendered by the detail process (or Table 4) for these particular water-sheds and the assumptions therefor and the rainfall curve used.

Values t_c and Q_c (Columns (6) and (7) of Table 9) are not intended to represent exactly the actual times and quantities for the respective water-sheds, which would obtain in an economically designed storm drain system constructed of closed conduit. If the assumptions of the rational method are strictly adhered to, it must be assumed that, in such systems, the storm which produces the quantity for a given water-shed, is not of sufficient intensity to fill the conduit above the outlet entirely. To allow for this condition with the formula through the use of Table 3 a higher average value of Factor F should be selected from Table 11 (Appendix II). However, the ratio, $\frac{Q_b}{Q_c}$, does represent closely the actual error which would occur in open channel work.

(7) It is interesting to compare Values t_a and t_1 (Columns (2) and (9) of Table 9). Both sets of values resulted from the detail "tabling" process, the former on the assumption of 10-min., and the latter of 20-min., initial inlet time. The difference in inlet times used is 10 min. If instead of using the values, t_1, this difference is added to the values, t_a, the error in Q is

measured by the ratio, $\frac{\sqrt{t_1}}{\sqrt{t_a + 10}}$ (since, for the rainfall curve used, Q varies inversely as the square root of t). This error, like the ratio, $\frac{Q_b}{Q_c}$, may be expected to vary somewhat for differences in inlet times assumed, water-sheds, and rainfall curves. It will be smaller (on the average) than $\frac{Q_b}{Q_c}$ for all ordinary additions.

(8) Remarks (6) and (7) indicate that in all cases Method No. 2 may be used to approximate run-off, the error in result being well within the automatic error due to the ordinary detail "tabling" process.

A better understanding of the reason for this small error may be had from the following illustration, which also indicates how greater refinement may be attained. Referring to the solution of Water-Shed No. 2 (see "Remark (2)"), $t_b = 43.4$ min.; and $Q_b = 972$ cu. ft. per sec. (10-min. inlet time).

If 20-min. inlet time has been used, the time would have been, approximately, $t_b + 10$, or 53.4 min., and the total run-off, by Equation (1), will be

$$Q = 0.4 \times \frac{9.69}{(53.4)^{0.5}} \times 1\,653 = 877 \text{ cu. ft. per sec.}$$

The maximum velocity (for this second consideration) on the average slope, and F, is: $V = \frac{F^2}{S^{\frac{1}{8}}} s^{\frac{1}{2}} Q^{\frac{1}{4}} = \frac{(6.63)^2}{(3.93)^{\frac{1}{8}}} (0.00393)^{\frac{1}{2}} (877)^{\frac{1}{4}} = 12.65$ ft. per sec. The average velocity equals $v = P\ V = 0.54 \times 12.65 = 6.83$. Finally, $t = \frac{L}{v} = \frac{18\,315}{6.83} = 2\,682$ sec. $= 44.7$ min.

This is an approximation of the time of concentration due to the smaller average quantity of water running. Furthermore, 44.7 + 10 min. = 54.7, which closely approximates the 54.9 min. resulting from the 20-min. inlet "tabling".

Reminder.—The solution of the run-off problem, through the medium of any assumptions, must be one of trial. The assignment of an inlet time (if used) and the run-off coefficient depend on the intensity. Since the intensity must be anticipated, a second solution may be necessary when the intensity varies sufficiently from the assumptions. It is hoped that future experimentation on the run-off coefficient will furnish more reliable data for selection.

APPENDIX I

Nomenclature for Part II

The following nomenclature, in brief, is used:

A = area of water-shed, in acres.
A' = area of initial area, or nucleus, in acres.
a = area of water-shed, in square feet.

$B = \sqrt{\frac{P}{L}}$.

C = coefficient representing the ratio of the rate of run-off to the rate of rainfall.

C' = coefficient of run-off for initial area, or nucleus.

c = constant in Chezy formula, $v = c\sqrt{r\,s}$.

e = a positive fractional exponent of time, t, in Equation (2).

f = correction factor (see Fig. 3).

F = constant, depending on shape and condition of main channel of flow; $F = N\,S^{\frac{1}{16}}$.

H = a given period of rain, in hours, $R_H = \frac{X}{H}$.

i = intensity of rainfall, in cubic feet per second per acre, or, approximately, in inches per hour.

i' = average intensity on initial area, or nucleus.

J = constant $= \frac{N}{r^{\frac{1}{4}}}$.

K = constant in Equation (2) $= i\,t^e$.

K' = constant.

K'' = constant.

l = any length, in feet.

L_1, L_2, L_3, etc. = subdivisions of l.

L = length, in feet, which water must traverse in running from the most remote portion of the water-shed to its outlet.

L' = length, in feet, which water must traverse in running from the most remote portion of the nucleus to its outlet.

M and M' = constants.

m = percentage of precipitation, i, being stored.

N = constant $= \frac{\left(c\sqrt{r}\right)^{\frac{1}{2}}}{Q^{\frac{1}{8}}} = \frac{F}{S^{\frac{1}{16}}}$.

P = constant, depending on shape of area of water-shed and its manner of concentration, $P = \frac{v}{V}$.

$P_1 = \frac{v'}{V'}$.

P' = constant, depending on beginning nucleus, = corrected P_1.

Q = rate of run-off, in cubic feet per second.

Q' = rate of run-off from nucleus, in cubic feet per second.

R_H = average rainfall intensity, in inches per hour, for a period of H hours, $R_H = \frac{X}{H}$.

r = hydraulic radius in $v = c\sqrt{r\,s}$; also, rate of retention.

s = slope, or fall, in feet per foot, in length of main channel of flow.

S = fall, in feet per 1 000 ft. of main channel of flow; $S = 1\,000\,s$.

t = time of concentration of water-shed = time required for the water to travel from the most remote portion of the water-shed to its outlet.

t' = time of concentration of nucleus.

v = average velocity of water, in feet per second, in traversing the length, L. $v = \frac{L}{t}$, or, hydraulically, $v = c\sqrt{r\,s}$.

v' = average velocity of water, in feet per second, in traversing the length, L'. $v' = \dfrac{L'}{t'_{\frac{s}{2}}}$.

V = velocity of Q at outlet of water-shed, based on average values of F and S for the water-shed.

V' = velocity of Q' based on average values of F and S for the water-shed.

$W = \dfrac{a}{L} = \dfrac{43\,560\,A}{L}$.

$W' = \dfrac{43\,560\,A'}{L'}$.

X = inches of rain for a period of H hours; $R_H = \dfrac{X}{H}$.

z = rate of absorption plus rate of evaporation, in cubic feet per second per acre.

APPENDIX II

Method of Calculating Values of N and F

Begin with, say, a 14 by 7-ft., open, rectangular channel with an area of 98 sq. ft., and an hydraulic radius of 3.50. Next, select from any reliable set of hydraulic tables (showing velocities for varying values of the hydraulic radius, Kutter's n, and the slope) the velocity for an hydraulic radius of 3.50 in the column of $s = 0.01$ (or $S = 10$), with Kutter's $n = 0.013$. This will be $V = 26.25$. Calculate $Q = 98 \times 26.25 = 2\,572.5$.

For $s = 0.01$, $c\sqrt{r} = 10$ $V = 262.5$, and $\left(c\sqrt{r}\right)^{\frac{1}{2}} = 16.20$. Substituting this value and the value, $Q = 2\,572.5$, in Equation (18), $N = 6.07$. This factor may be calculated for varying values of Kutter's n in the foregoing manner and placed on a line opposite an open rectangular channel, as in Table 10. This table was prepared by slide-rule computations for the purpose of illustrating the method.

It was shown that Equation (18) applies for a given shape and condition of channel. Having determined N for the 2 by 1 (width by height) open conduit, its value may be calculated for other shapes (for the small change in the hydraulic radius due to shape alone) by letting $N \propto r^{\frac{1}{4}}$. Thus, having assumed an area of 98 sq. ft for a 2 by 1 open conduit, calculate the relative r and $r^{\frac{1}{4}}$ for various shapes of channels using an area of 98 sq. ft., and arrange them in tabular form for reference. Then, since $N \propto r^{\frac{1}{4}}$, $N = J\,r^{\frac{1}{4}}$, or,

$$J = \frac{N}{r^{\frac{1}{4}}} \qquad (47)$$

the corresponding values of N for any shape of channel may be calculated, and the table expanded. For instance (using $N = 6.07$), $J = \dfrac{N}{r^{\frac{1}{4}}} = \dfrac{6.07}{1.366} = 4.44$.

Then, selecting any shape, such as circular pipe, with $r^{\frac{1}{4}} = 1.292$, $N = J\,r^{\frac{1}{4}} = 4.44 \times 1.292 = 5.74$.

The value of $J = 4.44$ may be used for calculating N for other shapes under Kutter's $n = 0.013$. In like manner the value of J obtained through the use of N, under other values of Kutter's n, may be used for calculating N for various shapes.

TABLE 10.—ILLUSTRATION OF METHOD OF CALCULATING VALUES OF N

Values of Kutter's n		.....	.....	0.011	0.013	0.015	0.020	0.025	0.030	0.035
Relative r, 4¼ (area 98)		r	$r^{\frac{1}{4}}$	N	N	N	N	N	N	N
Pipe		2.793	1.292	6.15	5.74	5.39	4.83	4.49	4.21	3.99
Single box (closed)	1 by 1	2.475	1.254	5.97	5.57	5.24	4.68	4.36	4.09	3.87
	2 by 1	2.333	1.236	5.89	5.49	5.15	4.61	4.30	4.03	3.81
	etc.	.....	.....	.....	.....	.....	.....	.....	.....	.....
Double box (closed)	1 by 1	1.750	1.150	5.47	5.11	4.80	4.29	4.00	3.75	3.55
	2 by 1	1.650	1.133	5.39	5.04	4.73	4.23	3.94	3.69	3.50
	etc.	.....	.....	.....	.....	.....	.....	.....	.....	.....
Triple box (closed)	1 by 1	1.428	1.093	5.20	4.86	4.56	4.08	3.80	3.56	3.38
	2 by 1	1.347	1.076	5.12	4.78	4.49	4.02	3.74	3.50	3.32
	etc.	.....	.....	.....	.....	.....	.....	.....	.....	.....
Open rectangular channel	2 by 1	3.50	1.366	6.50	6.07	5.70	5.10	4.75	4.45	4.22
	3 by 1	3.425	1.359	6.47	6.04	5.67	5.07	4.73	4.43	4.20
	1 by 1									
	4 by 1	3.305	1.347	6.41	5.98	5.63	5.03	4.68	4.39	4.16
	etc.	.....	.....	.....	.....	.....	.....	.....	.....	.....
	30 by 1	1.695	1.140	5.43	5.07	4.76	4.26	3.96	3.71	3.52

Having calculated the values of N for $S = 10$, values of F may be calculated. It was shown, in introducing Equation (21), that,

$$N = \frac{F}{S^{\frac{1}{16}}} \qquad (48)$$

Therefore, using $s = 10$, $N = \frac{F}{(10)^{\frac{1}{16}}}$, or $F = N\,(10)^{\frac{1}{16}}$.

Thus, having found the values of N under an assumed S, the factor, F, may be readily calculated and arranged as shown in Table 11. This table, showing values of F for channels of varying shape and condition, may be used in its present form, or may be expanded for use in general hydraulic work.

For N in Equation (18) substitute its value in Equation (48) and simplify, thus,

$$c\sqrt{r} = \frac{F^2}{S^{\frac{1}{8}}}\,Q^{\frac{1}{4}} \qquad (49)$$

In turn, substituting this value of $c\sqrt{r}$ in Equation (13): $v = c\sqrt{r\,s} = \frac{F^2}{S^{\frac{1}{8}}}\,s^{\frac{1}{2}}\,Q^{\frac{1}{4}} = 0.03162\,F^2\,S^{\frac{3}{8}}\,Q^{\frac{1}{4}}$, or,

$$v = 0.4217\,F^2\,s^{\frac{3}{8}}\,Q^{\frac{1}{4}} \qquad (50)$$

Thus, for a given run-off, Q, the approximate velocity may readily be computed for any definite shape with an assigned Kutter's n.

Since the velocity varies directly as F^2, Table 11 may also be used as a handy guide to the efficiency of relative shapes for a given Kutter's n.

Statement.—Heretofore, it has not been customary to allow for variation in the shape and condition of the main channel of flow in run-off formulas. The writers, however, have considered it well worth while to do so. In order to devise a practical working table of variations it was necessary to select, carefully, the velocity formula which could be used most advantageously for the purpose. The Kutter formula for c in $c\sqrt{r\,s}$ has been, and is at present, used very widely in hydraulic computations. It was chosen because of its general application to all shapes and conditions of channels. The exponential form for the combination of the Kutter and Chezy formulas (Equation (50)) affords a very practical means of arriving at a rational time of concentration (which is based on average velocities) of a given water-shed. Considering the inexact nature of the run-off problem, it is close enough for all practical purposes.

TABLE 11.—VALUES OF FACTOR F

Kutter's n			0.011	0.013	0.015	0.020	0.025	0.030	0.035	0.100
CLOSED SECTIONS										
Pipe			7.10	6.63	6.22	5.57	5.19	4.86	4.61	....
Closed semi-circular			6.85	6.38	6.00	5.36	5.00	4.68	4.44	....
Closed semi-elliptical			7.05	6.57	6.16	5.53	5.14	4.82	4.56	....
Flattened horseshoe			6.99	6.52	6.11	5.49	5.10	4.79	4.53	....
Rectangular (width by height)	Single	1 by 2	6.79	6.34	5.95	5.32	4.96	4.65	4.42	....
		1 by 1	6.89	6.43	6.05	5.40	5.04	4.73	4.47	....
		2 by 1	6.79	6.34	5.95	5.32	4.96	4.65	4.42	....
		3 by 1	6.65	6.21	5.83	5.21	4.86	4.55	4.32	....
		4 by 1	6.51	6.08	5.72	5.11	4.77	4.46	4.23	....
	Double	1 by 2	6.22	5.82	5.46	4.88	4.55	4.26	4.04	....
		1 by 1	6.32	5.90	5.54	4.96	4.62	4.33	4.10	....
		2 by 1	6.22	5.82	5.46	4.88	4.55	4.26	4.04	....
		3 by 1	6.10	5.68	5.34	4.77	4.46	4.17	3.95	....
	Triple	1 by 2	5.91	5.52	5.18	4.63	4.32	4.04	3.84	....
		1 by 1	6.00	5.61	5.27	4.71	4.39	4.11	3.90	....
		2 by 1	5.91	5.52	5.18	4.63	4.32	4.04	3.84	....
		3 by 1	5.80	5.42	5.08	4.55	4.24	3.97	3.77	....
OPEN SECTIONS										
Open semi-circular			7.74	7.21	6.77	6.05	5.64	5.28	5.01	3.38
Rectangular (width by height)	Single	1 by 2	7.10	6.64	6.24	5.58	5.20	4.86	4.61	3.11
		1 by 1	7.40	6.91	6.50	5.81	5.40	5.07	4.80	3.23
		2 by 1	7.51	7.01	6.58	5.89	5.48	5.14	4.87	3.28
		3 by 1	7.47	6.98	6.55	5.86	5.46	5.12	4.85	3.26
		4 by 1	7.40	6.91	6.50	5.81	5.40	5.07	4.80	3.23
		6 by 1	7.24	6.77	6.35	5.68	5.29	4.95	4.70	3.16
		8 by 1	7.10	6.64	6.24	5.58	5.20	4.86	4.61	3.11
		10 by 1	6.99	6.51	6.12	5.47	5.09	4.78	4.53	3.05
		12 by 1	6.87	6.42	6.03	5.39	5.02	4.70	4.46	3.00
		20 by 1	6.52	6.11	5.73	5.12	4.77	4.47	4.24	2.85
		30 by 1	6.27	5.86	5.50	4.92	4.57	4.28	4.06	2.74
		100 by 1	5.45	5.08	4.76	4.25	3.97	3.73	3.53	2.38

TABLE 11.—(*Continued*)

Kutter's n	0.011	0.013	0.015	0.020	0.025	0.030	0.035	0.100
Open Sections with Side Slopes								
Side slopes: Bottom width by height								
½ : 1 — V-shape	7.31	6.82	6.40	5.72	5.33	5.00	4.74	3.19
½ : 1 — 1 by 1	7.64	7.12	6.68	5.99	5.58	5.23	4.95	3.34
½ : 1 — 2 by 1	7.61	7.10	6.66	5.96	5.56	5.20	4.94	3.33
½ : 1 — 4 by 1	7.45	6.95	6.52	5.83	5.43	5.09	4.83	3.26
½ : 1 — 8 by 1	7.12	6.64	6.23	5.58	5.19	4.87	4.62	3.11
½ : 1 — 16 by 1	6.68	6.23	5.85	5.24	4.89	4.57	4.34	2.92
½ : 1 — 30 by 1	6.23	5.86	5.50	4.91	4.58	4.29	4.07	2.75
½ : 1 — 100 by 1	5.45	5.08	4.77	4.25	3.97	3.72	3.53	2.38
1 : 1 — V-shape	7.51	7.00	6.58	5.87	5.48	5.13	4.87	3.28
1 : 1 — 1 by 1	7.60	7.08	6.64	5.95	5.54	5.19	4.92	3.32
1 : 1 — 2 by 1	7.54	7.02	6.60	5.90	5.50	5.15	4.89	3.30
1 : 1 — 4 by 1	7.37	6.87	6.45	5.76	5.37	5.04	4.77	3.22
1 : 1 — 8 by 1	7.07	6.57	6.17	5.53	5.15	4.83	4.58	3.08
1 : 1 — 16 by 1	6.68	6.21	5.83	5.23	4.87	4.55	4.32	2.91
1 : 1 — 30 by 1	6.25	5.81	5.47	4.89	4.55	4.27	4.05	2.72
1 : 1 — 100 by 1	5.45	5.07	4.76	4.25	3.97	3.73	3.53	2.38
2 : 1 — V-shape	7.31	6.82	6.40	5.72	5.33	5.00	4.74	3.19
2 : 1 — 1 by 1	7.31	6.82	6.40	5.72	5.33	5.00	4.74	3.19
2 : 1 — 2 by 1	7.27	6.78	6.36	5.70	5.30	4.97	4.72	3.18
2 : 1 — 4 by 1	7.14	6.65	6.25	5.59	5.21	4.88	4.64	3.12
2 : 1 — 8 by 1	6.90	6.44	6.05	5.40	5.04	4.71	4.48	3.03
2 : 1 — 16 by 1	6.58	6.12	5.75	5.14	4.80	4.49	4.27	2.88
2 : 1 — 30 by 1	6.20	5.76	5.41	4.85	4.52	4.23	4.01	2.70
2 : 1 — 100 by 1	5.42	5.06	4.75	4.24	3.95	3.70	3.52	2.37
3 : 1 — V-shape	7.04	6.58	6.17	5.51	5.14	4.83	4.57	3.08
3 : 1 — 1 by 1	7.04	6.58	6.17	5.51	5.14	4.83	4.57	3.08
3 : 1 — 2 by 1	7.01	6.53	6.14	5.49	5.12	4.79	4.54	3.06
3 : 1 — 4 by 1	6.95	6.45	6.07	5.42	5.06	4.74	4.50	3.03
3 : 1 — 8 by 1	6.75	6.29	5.90	5.28	4.92	4.61	4.38	2.96
3 : 1 — 16 by 1	6.49	6.04	5.67	5.07	4.73	4.43	4.20	2.83
3 : 1 — 30 by 1	6.14	5.72	5.38	4.81	4.49	4.20	3.99	2.68
3 : 1 — 100 by 1	5.40	5.03	4.72	4.22	3.93	3.69	3.50	2.36

Some more recent experimentation has been done, which indicates that Kutter's formula is somewhat in error for certain shapes of section. The Scobey formula for concrete pipe flowing full is one of the best known. This formula is:

$$Q = 0.00546\ C_s\ d^{2.625}\ H^{0.5} \qquad (51)$$

or,

$$v = C_s\ d^{0.625}\ H^{0.5} \qquad (52)$$

in which, Q is the discharge, in cubic feet per second; v, the velocity, in feet per second; C_s, Scobey's coefficient; d, the diameter, in inches; and H, the friction head per 1 000 ft.

For comparative purposes it may be of interest to reduce the Scobey formula, showing the velocity in terms of C_s, H, and Q. From Equation (51),

$d^{2.625} = \dfrac{Q}{0.00546\ C_s\ H^{0.5}}$, and,

$$d^{0.625} = \frac{Q^{0.2381}}{0.2892\ C_s^{0.2381}\ H^{0.119}} \qquad (53)$$

Substituting Equation (53) in Equation (52),

$$v = 3.549\ C_s^{0.7619}\ H^{0.381}\ Q^{0.2381} \quad \ldots\ldots(54)$$

Using the nomenclature of this paper, and replacing H in Equation (54) by S,

$$v = 3.459\ C_s^{0.7619}\ S^{0.381}\ Q^{0.2381} \quad \ldots\ldots(55)$$

Comparing Equation (55) (which is the Scobey formula reduced to like terms) with the exponential form of the Kutter-Chezy formula (Equation (50)), it may be seen that the exponents of S and Q do not differ greatly.

This difference will be of less consequence in its effect on run-off calculations, as may be shown by substituting the value for v, shown in Equation (55), in the formula, and following the same method of reduction as outlined in connection with Equation (14):

$$Q = (3\ 600\ H)^e\ C\ A\ R_H\ \frac{v^e}{L^e} \quad \ldots\ldots(56)$$

Factor H in Equation (56) is not to be confused with the H in the Scobey formula (Equation (51)), which was replaced by S.

For v in Equation (56) substitute Pv, in which, v is given in Equation (55); thus

$$Q = (3\ 600\ H)^e\ C\ A\ R_H\ \frac{[P\ (3.459\ C_s^{0.7619}\ Q^{0.2381}\ S^{0.381})]^e}{L^e}$$

or,

$$Q = (12\ 452.4\ H)^{eq} \left(\frac{P}{L}\right)^{eq} (C\ A\ R_H)^q\ C_s^{0.7619\,eq}\ S^{0.381\,eq} \quad \ldots\ldots(57)$$

in which, q has been substituted for the exponent, $\frac{1}{1 - 0.2381^e}$, for simplicity.

Equation (57) may be compared with Equation (24). It may be reduced to a special form by letting $e = 0.5$, in which case, $q = \frac{1}{1 - 0.1191} = 1.1352$, and,

$$Q = 210.9\ H^{0.5675} \left(\frac{P}{L}\right)^{0.5675} (C\ A\ R_H)^{1.135}\ C_s^{0.4324}\ S^{0.2162} \quad \ldots\ldots(58)$$

Substituting $\sqrt{\frac{P}{L}} = B_1$ and $\sqrt{S} = D$,

$$Q = 210.9\ H^{0.5675}\ (C\ A\ R_H\ B_1)^{1.135}\ (C_s\ D)^{0.4324} \quad \ldots\ldots(59)$$

and when $H = 1$ hour,

$$Q = 210.9\ (C\ A\ R_1\ B_1)^{1.135}\ (C_s\ D)^{0.4324} \quad \ldots\ldots(60)$$

Equations (58), (59), and (60) may be compared with Equations (27), (28), and (29), respectively. It may be noted that the difference in exponents is small and that a certain compensating effect takes place between the exponents for S and those for the other factors.

Equations (58), (59), and (60) may be used for run-off calculations by means of C_s values established by F. C. Scobey, M. Am. Soc. C. E. It is understood by the writers, however, that the Scobey formula for velocity

is restricted to circular pipe flowing full. In this case Equations (57) to (60), inclusive, resulting from its use, are likewise restricted.

To illustrate further the direct applicability and correctness of the factors and formulas herein advocated, they are applied to the upper part of the Compton Creek water-shed, which lies immediately south of the City of Los Angeles. This area is divided into two main parts, the drainage channels of which converge near the lower end of the area. (Somewhat similar to Type 17, Fig. 2.) The western part of this area contains 16 731 acres and the eastern part, 12 361 acres.

Since the topography of both sides is similar, the West Channel, with a length of 64 600 ft., will control the time of concentration, since it is the longer.

Example 11.—Find the approximate time of concentration using a rainfall curve of the type of Equation (3).

Let $R_1 = 1.50$ in. per hour.

$S = 2.07$ ft. per 1 000 ft. from U. S. Topographical map. (Total fall, 134 ft.)

$F = 6.5$ (selected because part of the channel will be a closed rectangular box section running part full, and the remainder, pipe).

$$\frac{L}{W} = \frac{(64\,600)^2}{43\,560 \times 16\,731} = 5.7.$$

Using Table 3, and Method No. 2, $P = 0.47$. Let $C = 0.4$; then $Q = 6\,466$ cu. ft. per sec. by Equation (29); $i = 0.966$ by Equation (1); and $t = 144.7$ min. by Equation (26).

Example 12.—Find the approximate time of concentration which would result from the usual tabling process, with all data the same as Example 11 except that, $P = 0.55$ (selected from Table 4, Method No. 2); and, $F = 6.0$ (assuming all conduits running full). Then, $Q = 6\,377$ cu. ft. per sec. by Equation (29); $i = 0.953$ by Equation (1); and, $t = 148.7$ min. by Equation (26).

Example 13.—Find the approximate time of concentration, using a rainfall curve of the type, $i = \frac{K}{t^{0.7}}$ and $C = 0.35$. All other data are the same as Example 11.

This solution could be made by the use of Equation (25) with proper exponents, using $e = 0.7$, but a close approximation may be made as follows: Using the time, 144.7 min. (Example 11), read the intensity from a curve of $i = \frac{K}{t^{0.7}}$ or calculate: $i = 0.81$ (see Equation (45)); $Q = 0.35 \times 0.81 \times 16\,731 = 4\,743$ cu. ft. per sec. by Equation (1); $V = 14.56$; $v = P\,V = 0.47 \times 14.56 = 6.84$; and $t = \frac{L}{v} = \frac{64\,600}{6.84} = 157.3$ min.

The solution by means of Equation (25) gives $t = 160$ min. The difference between the times, 144.7 and 157.3, is due, of course, to the smaller average quantity of water running. In the same manner, approximations of the time of concentration may be found for any curve in use.

Example 14.—Find the approximate time of concentration of all data the same as Example 11 except that $C = 0.30$ and $F = 4.72$ (from Table 11, main channel, open earth, Kutter's $n = 0.035$, the bottom width is twice the depth, and the side slopes are 2 : 1.) Then, $Q = 3\,228$ cu. ft. per sec. by Equation (29); $i = 0.643$ by Equation (1); and, $t = 326$ min. by Equation (26).

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DISCUSSIONS

PLASTIC FLOW IN CONCRETE ARCHES

Discussion

BY LORENZ G. STRAUB, JUN. AM. SOC. C. E.

LORENZ G. STRAUB,[1] JUN. AM. SOC. C. E. (by letter).[1a]—Much information and constructive criticism has been offered in the generous response of discussion by members of the Engineering Profession. However, despite the earnest effort to point out and distinguish between real and hypothetical considerations, there remained a certain amount of confusion in some instances.

There is a general expression of the viewpoint that much remains to be discovered regarding the time-stress-strain relations of concrete. Some of those who discussed the paper believe the separation of deformation into elastic and inelastic deformation is without justification.

Mr. Jakobsen calls attention to valuable information concerning the stress-strain relations for plain concrete. However, the writer does not feel that "the separation of elastic and plastic deformations seems an artificial procedure". It is true that this is a difficult segregation and that most "tests show the combined deformation only", but it is doubtful in the case of concrete whether "the plastic deformations disappear in due course of time, and are not in any sense permanent deformations, except when the elastic limit has been exceeded". If so, then concrete must be considered to pass the elastic limit at a stress far below the normal working stress; in any case, experience indicates continuous deformation to occur in concrete when subjected to normal loading conditions. More experimental data would be very valuable in this connection.

As pointed out by Mr. Jakobsen if the stress-strain curve is linear for plastic as well as for elastic deformation, Hooke's law applies, and the time factor effects the deformations but not the load stresses. In this particular case, if no account were to be taken of the effect of time-yield in connection with temperature deformation, shrinkage, yielding supports, etc., the ordinary elastic theory would suffice completely for making a stress analysis of the

NOTE.—Author's closure. The paper by Lorenz G. Straub, Jun. Am. Soc. C. E., was published in January, 1930, *Proceedings*. Discussion of the paper has appeared in *Proceedings* as follows: April, 1930, by Messrs. B. F. Jakobsen and C. A. P. Turner; May, 1930, by Messrs. Edward Godfrey, J. H. Griffith, and Charles W. Comstock; August, 1930, by Messrs. A. A. Eremin, Josef Melan, Fredrik Vogt and H. J. Gilkey, and E. Probst; September, 1930, by Paul Bauman, M. Am. Soc. C. E.; and December, 1930, by Messrs. G. C. Staehle, and A. Floris.

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[1a] Received by the Secretary, February 20, 1931.

structures; but even for the hypothetical case in which the stress-strain curve is linear for both plastic and elastic deformation, an analysis of the stresses incurred as a result of temperature changes, yielding supports, etc., requires that distinction be made between plastic deformation and elastic deformation.

The writer is in full accord with Mr. Jakobsen's statement that the exponential law for elastic deformation will not fully explain the somewhat perplexing problem of determining why plain concrete beams fail at such high computed stresses, although the computations based upon the law indicate a variation in the right direction. It is highly probable that more accurate information concerning the physical properties will shed more light upon this subject.

In connection with his statement that the writer "apparently computes the plastic deflection of a 140-ft. arch as 3 in., * * *," it seems that Mr. Turner has not clearly understood the diagram in question. Fig. 30 concerns a very hypothetical case in which the ratio, $\frac{H_{DL}}{H_{RS}}$, is considered equal to 25. Under such circumstances, the line of thrust will be eccentric with respect to its normal position. It was considered, therefore, that the line of thrust was not allowed to return to its normal position as a result of plastic flow, but instead was maintained eccentric in consequence of continuous contraction of the arch rib (temperature change), resulting in progressive angle changes within the arch rib. Obviously, this is merely a simple computation made for the purpose of showing that, for the particular hypothetical case, continued deformation might occur without change in the stress distribution in the arch rib. Computations for the case in which all deformation subsequent to removal of forms is due to plastic flow would indicate a crown deflection of only a fraction of the amount referred to by Mr. Turner, most of which would take place in the first few months. There is no question that a certain amount of rib-shortening and, consequently, deflection of the crown, occurs progressively at least during the early life of an arch. This is particularly emphasized by the result of rib-compensation methods (hydraulic jacks operated at the crown of arches during the early life of the structures), used in France[2] and Germany. The results of experimental investigation[3] on arches by Wilbur M. Wilson, M. Am. Soc. C. E., also show continued deformation incurred under constant loading conditions.

The writer is not in a position to say whether "Messrs. E. B. Smith, F. R. McMillan, and A. H. Fuller and C. C. Moore were mistaken in supposing they had discovered a general phenomenon of plastic flow in concrete as applied to the practical structure". Nevertheless, he does not doubt that they observed the phenomena of progressive deformation, and more recently that Raymond E. Davis, M. Am. Soc. C. E., and G. E. Troxell,[4] Assoc. M. Am. Soc. C. E., Mr. Oscar Faber,[5] and others observed that progressive deformation takes place in concrete over a period of many months under the influence of

[2] M. Freyssinet, *Le Genie Civil*, Vol. 79, p. 126, 1921.
[3] *Bulletin No. 202*, Eng. Experimental Station, Univ. of Illinois, 1930, p. 96.
[4] *Proceedings*, Am. Soc. for Testing Materials, 1929, p. 696.
[5] *Paper No. 4645*, Inst. C. E., London.

a continuously applied load. Mr. Turner's belief that a concrete arch having a cross-sectional area of 72 by 100 in. is very different from that of the ordinary test experiments, deserves careful consideration, because any such deviations are of great general importance in connection with experimental investigations of concrete.

Mr. Godfrey's timely emphasis of the psychological danger involved in the application of complex theories is worthy of special notice. Far too frequently fine-spun, precise analyses are worse than useless because of the false sense of security that they arouse in the design. Although he is no advocate of methodology in structural design, the writer desires to call attention to the noteworthy innovations in procedure for analyzing statically indeterminate structures which have recently been set forth by Hardy Cross, M. Am. Soc. C. E.[6] These should go a long way toward overcoming the mechanical methods of applying the theory of elasticity.

The concise, scholarly discussion by Professor Griffith considers many additional factors and does much to clarify the confused viewpoints in the interpretation of the phenomena of elasticity, plasticity, and viscosity. Professor Griffith's arguments make it seem reasonable that concrete should be called a "visco-elastic" material and that the occurrence of time-yield might possibly better be called viscous flow rather than plastic flow. It is only reasonable that technologists and scientists should meet on a common ground as far as definition is concerned and, therefore, the remarks contained in this discussion are especially important.

The writer does not consider the skepticism held by Mr. Comstock entirely justifiable. It is quite true that shear deformations result in plane sections not remaining exactly plane after flexural deformations occur. In the case of the arch, however, as pointed out in the paper (see "General Relations; Proposed Theory of Elasticity for Arches"),

"Since shear deformations have only slight effect on the change in conformation of a loaded arch rib in comparison to the effects of moment and rib-shortening deformations, the deformations due to shear will be neglected in the analysis."

It is fully agreed that in the case of pure flexures there will be a certain amount of approximation resulting from neglecting shear deformations, just as in the case of the usual approximate procedure of applying the ordinary theory of elasticity to beams. However, in the case of properly designed concrete arch ribs, the error introduced is unquestionably negligible.

Considering fixed-ended concrete arches, the writer cannot agree with Mr. Comstock's statement that "* * * it is of no importance whether or not the whole or any part of the strain disappears on removal of the applied forces". Where the deformation is progressive and dependent upon the concentration of stress, there results (in the case of arches and similar structures) a gradual displacement of the line of thrust which would not occur in the case of a perfectly elastic material. In this connection the writer wishes to emphasize the fact that, in order to make a satisfactory analysis where time-yield occurs, the entire structure must be taken into account and not merely its parts. In answer to Mr. Comstock's question why 1.3 was selected for the

[6] *Proceedings*, Am. Soc. C. E., May, 1930, Papers and Discussions, p. 919.

value of m, the writer wishes to refer to the paragraph following Table 3 in the paper:

"It is realized, of course, from the experimental data on the elastic properties of concrete that a variable relation of stress to elastic strain in which m is as high as 1.3 is possibly never obtained in practice. However, an extreme value of m has been chosen in order that the results may show characteristic trends more prominently. Also, since a value of p of 1.3 in the proposed theory of plasticity is probably about correct for concrete 3 or 4 months old, this value makes a convenient starting point for an analysis in which the plastic properties of the material are considered".

The paper repeatedly emphasizes the need for more experimental data before a conclusive set of values can be chosen for the parameters in the proposed equations.

Furthermore, with reference to Mr. Comstock's statement that, "* * * it is not clear on what ground the author justifies the assumption that it [strain] may be expressed as the product of two functions, one of stress only and the other of time alone", the writer refers to Figs. 14 and 15. It will be observed that both these series of experiments indicate that the deformation is very well expressed by independent functions of stress and of time.

Unfortunately, not enough "time-yield" experiments extending over long periods are available at present, and the writer concurs in the opinion of Mr. Comstock that the value of m does vary with the age of the structure. Not only is this true, but the values of k, K, p, and q are also likely to vary with the age of the concrete, the latter even to a greater extent than m. No one questions that the modulus of elasticity varies with the age. However, the writer does not feel that this disproves the validity of the equations proposed. Extended experimental data may indicate—and the writer believes that this is not only possible but probable—that the parameters of the equations must be varied at intervals as the age of the structure increases in order to obtain satisfactory results, and it is possible that these variations might be quite large. In any case (although it is agreed that there is not sufficient experimental data to justify completely the general adoption of a stress-strain-time relation for concrete and that sometimes such studies may lead to preconceived notions), the writer feels that in not formulating reasonable relations based upon pioneer experimental studies an investigator is gathering information blindly without a definite purpose in view and, therefore, without recording all the facts desirable. This is true of much experimental data now available.

Considerable interesting and valuable information bearing upon the subject of this paper is brought to light by Mr. Eremin. Nevertheless, partial exception is taken to one of his statements referring to Fig. 21. If the properties of the material of construction are of a nature similar to those found from the experimental data available, there will be no appreciable redistribution of stresses in spite of the fact that excessive deflection occurs. It is true, however, that if the concrete at a particular point undergoes excessive yield similar to the condition which is likely to occur in the case of a steel beam, there would be a redistribution of stresses. Under such circumstances the effect would be similar to that of a settling support. However, although the center of a concrete beam may deflect progressively an excessive amount as a result

of time-yield, this does not indicate a redistribution of stresses; on the contrary, the stresses are likely to remain practically unchanged.

The remarks by Dr. Melan and also those of Dr. Probst are of particular interest because of the broad experience of these men in connection with the design, construction, and experimental studies of arches in their respective countries. Dr. Melan's exception to the writer's conclusion regarding the early removal of forms is doubtless well founded and unquestionably such a procedure should not be adopted without careful precautions.

Dr. Vogt and Professor Gilkey rightfully emphasize the need of more experimental data before an acceptable method of analysis, such as proposed by the writer, can be adopted. It is well to call attention to the fact that the properties of concrete vary with the age. Consequently, as is evident from a comparison of data now available, the values of the parameters in the exponential equations, particularly the value of q, vary greatly with the age of the concrete. For a very green concrete the value of q is quite low, while for an older concrete, q is larger by several hundred per cent. Consequently, the writer does not recommend that a single value of q be adopted for concrete for a long period of time. Just what values should be used will depend upon information obtainable only by further experimental work. It is agreed that the writer probably made his analysis cover a longer period of time than should have been assumed for the experimental data available. These studies were intended only to indicate the trends rather than to present numerically accurate results.

The writer does not feel satisfied with Equation (117), presented by Dr. Vogt and Professor Gilkey, for use in studies of indeterminate structures such as concrete arches. It is true that this relation apparently indicates the approximate relation of stress to total strain, but the equation is not sufficient to prove the writer's "Second Conclusion" that,

"If there is an external disturbance * * *, such as the settlement of a support, the plastic flow causes a gradual redistribution of stresses such that the structure approaches (but never reaches) its original state of equilibrium (no settlement of support)".

The statement quoted requires the further verification of the fact that as a result of time-yield the line of thrust again approaches its original position. Equation (117) in itself does not show this.

Even if the value, m, is greater than p—a condition which is conceivable, but not probable—certain advantages are given by the early removal of the arch centering, although unquestionably certain disadvantages are also introduced. For example, if at the time the centering is removed there is a certain amount of displacement of the piers, a re-adjustment will take place tending to cause the arch to approach its original state of equilibrium (no movement of supports); on the other hand, if m is greater than p, the disadvantage incurred is that the line of thrust will be pulled a short distance farther away from the center line of the arch rib than if m is equal to or less than p.

Very interesting opinions and further data are given by Messrs. Bauman, Staehle, and Floris. It is to be hoped that more long-time experiments will be performed, in order to give a more accurate knowledge of many of the physical properties of concrete and their variation with the age of the material.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

RELATION BETWEEN RAIL AND WATERWAY TRANSPORTATION: A SYMPOSIUM

Discussion

BY MESSRS. T. Q. ASHBURN, L. D. CORNISH, AND E. A. HADLEY

T. Q. ASHBURN,[5] ESQ. (by letter).[5a]—Too much has been said in the various discussions of this paper about the relative merits of rail, water, and motor transportation and the effect of such competition upon the revenues of the various carriers, and far too little about the interest of the public in securing the best transportation at the cheapest possible rate. Every citizen is vitally interested in the cost of transportation because it is concealed in the price of every article he utilizes, the food he eats, and the clothes he wears.

Every new country passes through four stages of transportation: Water; paths and roads; rail; and, finally, the one through which this country is now passing—that of co-ordination and co-operation.

Each of these stages follows as naturally as day follows night; each performs its function satisfactorily considering the condition involved; and each, in the order named, inevitably leads to the other. As soon as a better and cheaper, and more satisfactory, form appears, the less satisfactory, and the more expensive, form fights constantly to prevent the people of the United States from enjoying such transportation, and always on the ground that:

"The existing forms of transportation fill all our needs, and if you allow competition it will destroy our revenue. We are too important to be destroyed or hurt; we have expanded at the peoples demand, and it is unfair competition to subsidize some other form of transportation to our disadvantage."

It will be observed that there is no thought anywhere, in this argument, that the people who have "paid and paid" to get transportation, who have allowed one form to supersede another (and always by subsidies), have any

NOTE.—Authors' closures. Symposium published in March, 1930, *Proceedings*. Discussion of this Symposium has appeared in *Proceedings*, as follows: April, 1930, by Lewis C. Sorrell, Esq.; May, 1930, by Messrs. R. B. Kittredge, Harry A. Palmer, and J. Parker Snow; August, 1930, by Messrs. Baxter L. Brown and Theodore Brent; September, 1930, by Samuel O. Dunn, Esq.; December, 1930, by Henry W. Hobbs, M. Am. Soc. C. E.; and January, 1931, by C. W. Kutz, M. Am. Soc. C. E.

[5] Maj.-Gen., U. S. A.; Chairman and Executive, Inland Waterways Corporation, Washington, D. C.

[5a] Received by the Secretary, January 21, 1931.

legal, moral, or ethical right to patronize that form of transportation which is of most value to them.

The problem confronting the people of the United States is this: They have nearly $1 500 000 000 invested in harbors and navigable waterways, and they have been appropriating annually vast sums of money to continue making navigable streams, in the hope that a cheaper means of transportation will be created whereby the entire country will profit. This vast scheme has only been partly completed, and it will require a comparatively small sum to finish that part of it which has been found to be in accordance with sound economics and sound engineering principles. On its completion the people may look with certainty to the fact that cheaper and better co-ordinated water-rail-motor service will result in a system of transportation superior to any single system, affording every one a real return on the money which he has invested in the form of taxes.

These savings inherent in water transportation will not only be available to those communities fortunately located on navigable streams, because when this co-ordinated system of joint routes and rates is fully developed in accordance with the Denison Act, the shipper in the interior will receive the same savings in cents per hundred pounds through joint rates, as the man who lives near the river. The general principle of determining rates is to apply the water saving between the river ports to the all-rail rate between the points of origin and of destination, and then to establish a joint route, *via* the Federal Barge Lines, by rail and water, or by motor and water, cheaper than the all-rail or all-motor rate. For example, if the all-rail rate on a given commodity from Dubuque, Iowa, to New Orleans, La., were $6.00 per ton, the all-water rate normally would be $4.80 per ton (80% of the all-rail rate), a saving of $1.20. Now, if the all-rail rate from Wightman, Iowa, to New Orleans, for example, were $7.00 per ton, a joint route and rate by rail would be established from Wightman to Dubuque, by water to New Orleans, and the joint rate from Wightman to New Orleans would be $7.00 minus $1.20 (the water saving), which equals $5.80 per ton. The Wightman shipper would thus receive exactly the same savings in cents per hundred pounds as the Dubuque shipper.

This is exactly what the Inland Waterways Corporation proposes to do for every interior shipper; and yet the opponents of waterway development and of the operations of the Federal Barge Lines propose to abandon this vast network of waterways already partly completed, charge it to profit and loss, and admit that the entire policy of 110 years has been a colossal failure, instead of spending the money necessary to complete the system. It is submitted that no business organization, no combination of capital, would abandon any such sum as the $1 500 000 000 already spent to finish seven-eighths of a project which, until completion, is of little value, when by the appropriation of the additional 12½%, a reasonable return on the investment would become available.

In an endeavor to becloud the issue, and by using the same tactics that administrators of canals and highways used against railways in their early pioneering, a concerted attack is being made on the plan of the Federal Barge Lines to utilize streams, on the grounds that this organization is subsidized.

It is an agency created by Congress to demonstrate that water transportation is feasible, and economically sound, and it is the only agency available for such demonstration.

Although the Federal Barge Lines have had a net operating income since the inception of the Inland Waterways Corporation, although no money has been appropriated for anything except expansions of the line, although the Corporation could exist and operate and furnish cheap transportation indefinitely on its net operating income, tables such as that in Fig. 10[6] are constantly being published to lead the public to believe that it is being mulcted. It is well to note that the Inland Waterways Corporation has existed only since the latter part of 1924; yet these charges apply to the total expenditures on the waterways on which the Corporation operates, or expects to operate.

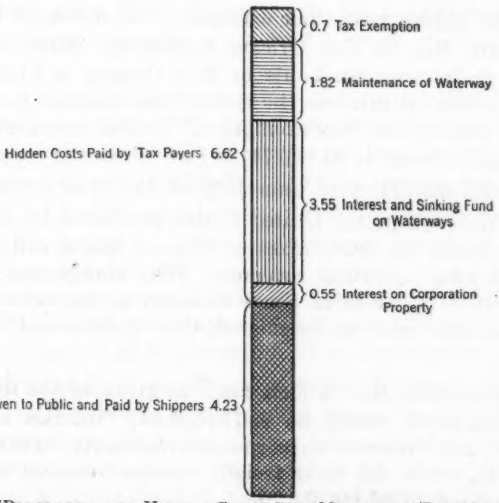
0.7 Tax Exemption
1.82 Maintenance of Waterway
Hidden Costs Paid by Tax Payers 6.62
3.55 Interest and Sinking Fund on Waterways
0.55 Interest on Corporation Property
Cost as given to Public and Paid by Shippers 4.23

FIG. 10.—"DISCLOSED AND HIDDEN COSTS," IN MILLS PER TON-MILE, OF INLAND WATERWAYS CORPORATION'S OPERATIONS

These attempts to prove that there are certain "hidden charges" which the citizen pays when he ships by the Federal Barge Lines, are only efforts to detract the shippers' attention from one important fact; namely, that he is actually getting cheaper transportation by water.

Certainly the people have been taxed to create and maintain their navigable streams, harbors, lighthouses, etc., to accomplish a certain definite purpose; and since they have been taxed once to create them, why, in order to bolster up a case of "hidden costs", should they be charged again with "tax exemption," "maintenance of waterway," "interest and sinking fund on waterway," and "interest on the Corporation [their own] property?"

The Inland Waterways Corporation is the agent created by Congress to prove that having thus taxed themselves, the people can get that for which they have been taxed, namely, cheaper transportation by water. This is some-

[6] From *New York Times*, December 28, 1930.

thing they had been unable to secure for years, because water transportation has been destroyed by the railroads; and to tax the Corporation, in its pioneering, demonstrative work with a preposterous system of charges which do not exist, and which the people have willed should not exist, is to deny to the people the right to reap the benefit of the investment made.

Assuming, however, that what opponents claim should be charged against water transportation in Fig. 10, are actually things which should be considered, certain errors and discrepancies may be pointed out so that the reader may judge whether or not they make a legitimate case.

The diagram (Fig. 10) was evidently prepared from data contained in the Annual Report submitted by the writer in 1928. It claims a "hidden cost" to the taxpayer of 0.7 mill per ton-mile on the ground of "tax exemption." The distance from Minneapolis, Minn., to New Orleans is 1 839 miles. According to Fig. 10 the hidden cost, "tax exemption" on a ton of freight transported from St. Louis, Mo., to New Orleans is $0.8078. Since the freight charged on a ton of grain from St. Louis to New Orleans is $1.60, the "hidden tax exemption" is 50% as great as the cost, if the diagram is correct.

The total corporation "tax exemption," hidden cost, given in this diagram for 1928, would amount to $1 066 741.90; or (since the freight revenue for the year was $6 445 353.70) more than 16% of the total revenue. According to statistics of railways in the United States, published by the Interstate Commerce Commission for 1928, taxes of Class 1 steam railways were equal to 6.37% of the total operating revenues. Why charge the Inland Waterways Corporation 16 + %, or three times as much as the railways? On the same basis that railroad taxes are calculated, this "hidden cost" would be 0.23 mill per ton-mile.

On the same basis, the "hidden cost," as given by the diagram of "maintenance of waterways," would be $2 773 528.94; "interest and sinking fund," $5 409 905.35; and "interest on corporation property," $838 154.35 (a total of $10 088 330.54), while the total freight revenue received by the Corporation actually amounted to $6 445 353.70.

Why is this item, "maintenance of waterways," charged to the Inland Waterways Corporation? According to a pamphlet issued by the U. S. Department of Commerce there are 200 common carriers, 98 contract carriers, and 187 private carriers, with a total value of $150 000 000, operating on the inland waterways of the United States. In 1928, to which year Fig. 10 evidently referred, the value of the Inland Waterways Corporation was approximately $16 000 000, or 10.66% of the total invested on inland waterways. A proportionate share, therefore (if such charge is legitimate), would be 0.19 mill per ton-mile for maintenance of waterways, instead of 1.82 mills per ton-mile. The interest and sinking fund on waterways (if such charge be legitimate) would be 0.378 mill per ton-mile.

"Interest on the corporation property" is here computed at 5.5% of the actual value at the time. As a matter of fact the Government can get all the money it wants at 4%, so that the interest would amount to only 0.4 mill per ton-mile. Now, add to the actual items, the "disclosed costs" (which included a net income of $327 712.30), and all that Fig. 10 purports to give as hidden

costs, in order to obtain the final cost per ton-mile. Then the results are, as follows:

Item	Mills per ton-mile
Tax exemption	0.26
Maintenance of waterway	0.19
Interest on sinking fund, etc.	0.38
Interest on Corporation property	0.4
Cost paid for transportation	4.23
Total	5.46

This may be compared with the 10.85 mills per ton-mile in Fig. 10. In comparison, the actual operating cost of the railroads equals almost exactly 10 mills per ton-mile, without considering any "hidden costs" in the form of land grants to the railways since 1850, amounting to 132 173 224 acres, or of the subsidy of the $40 000 000 per annum received through Pullman surcharges, or other items.

This "land grant" by the Government to the railroads is a continuing subsidy, since in the fiscal year, 1930, there were 62 249.29 acres of land certified or patented on account of railroad grants by the United States to the railroads precisely as they were certified or patented (or granted) in the early days. One cannot be certain exactly what these 132 000 000 acres, more or less, were sold for, but according to an advertisement issued by the Northern Pacific Railroad Company in 1871, the lands granted to the Union Pacific sold for $4.46 per acre; the school lands of Minnesota, $6.30 per acre; and the lands of the Illinois Central grant at $11.00 per acre, and the advertisement quotes an "average of $4.00 per acre." The Government price per acre at this time was given as $2.50, so that, taking this extremely low price, there is a "hidden cost" to the people of the United States of 132 000 000 acres at $2.50 per acre, or $330 000 000, given to the railroads to secure a "right of way," just as the money spent for rivers was given by the people to the people themselves to secure a "river right of way." Furthermore, the rights of way thus given to the railroads by the Government are carried in the value of the railroads, on which the people are requested to pay an additional 5½% return.

The United States Government spends millions of dollars on harbor improvements, lighthouse service and construction, building jetties, dredging bars, and keeping channels clear and open, in order to aid ocean transportation and make possible the delivery by the railroads of multiplied train loads of freight and passengers. Without this expenditure the seaport terminals of the railroads would be dead ends and would soon become worthless. Of course, the Government is not meeting this great annual outlay specifically for the railroads, but as a means to an end, and to build up, protect, and stimulate the transportation service of the entire country and, also, in the interest of all the people, and it is manifestly a good policy. The people need transportation and they develop it through one form or another of subsidy.

They are entitled to the best form of transportation available at the cheapest possible rate, and if they desire to pay a part of the cost of transportation by highway, or by inland waterways, or by granting to the railways the present

existing $40 000 000 subsidy of a surcharge on a Pullman ticket, they have a perfect right so to do. It would appear to be a much better policy on the part of all concerned not to attempt to mislead the people about hidden costs, but to furnish the best and cheapest possible transportation, in fair competition; and that system will come about through proper co-ordination and co-operation of all forms of transportation. It cannot be brought about by crying aloud that one form of transportation will destroy another, or by bickerings, evasions of self-evident truths, or unfair propaganda.

L. D. CORNISH,[7] M. AM. SOC. C. E. (by letter).[7a]—The writer wishes to record his appreciation of the very able discussion accorded his paper. Mr. Hadley's argument that inland waterway operation will not lower rail rates because of the fixed differential, is true only with respect to waterway common carriers regulated by the Interstate Commerce Commission. The competition of contract and private carriers on the Ohio and Mississippi Rivers has resulted, during 1930, in certain railroads applying for reduced rates on steel products from the Pittsburgh District. Contract and private carriers unregulated as to rates also constitute a real menace to the development and existence of the waterway common carrier because of their rate-cutting methods.

Professor Sorrell's discussion the writer considers as a conspicuous accomplishment of the main purpose of his paper, namely, to invite or provoke discussions of the subject by trained and experienced students of the transportation problem. His discussion of the various forecasts of future traffic demand and the methods used should be valuable to those interested in that phase of the subject. It appears that the writer's forecast of 870 000 000 000 ton-miles is too large even if it be considered as the shippers' demand on rail, truck, and inland water transportation agencies exclusive of the Great Lakes. This consideration becomes necessary on account of the phenomenal growth of truck traffic since 1925. Truck traffic constitutes a serious menace to the future welfare of the railroads. They must meet this competition by reducing their short-haul and terminal costs and the consequential freight rates and must provide store-door delivery for L. C. L. freight.

Railroad officials and railroad periodicals lay much emphasis on the fact that railroad companies pay property taxes while truck carriers have the tax-free use of highways built and maintained at public expense. It is surprising, indeed, that they apparently overlook the tax on gasoline and its effect on truck operation costs. Every State in the Union taxes gasoline for motor vehicle consumption. The tax ranges from 2 to 7 cents per gal., and probably averages about 3 cents. Heavy motor trucks consume about $\frac{1}{3}$ gal. per cargo ton-mile, and hence their gasoline tax for use of highways is about 6 mills per ton-mile, which is nine times the property tax and more than four times the maintenance costs paid by the railroad companies.

The suggestion of Professor Sorrell of a method of arriving at the probable social cost of a ton-mile of waterway transportation and the net social saving is very interesting. It would be an interesting subject for a useful thesis of a

[7] Chf. Engr., Div. of Waterways, State of Illinois, Chicago, Ill.

[7a] Received by the Secretary, February 14, 1931.

graduate student of economics. The writer feels indebted to Professor Sorrell for his splendid contribution to the subject and especially for his constructive criticism and discussion of the writer's paper.

Professor Kittredge charges the writer with two errors of major importance. First, that "figures which cover the total cost of railway transportation have been used in comparison with other figures which include only a part of the cost of waterway transportation".

This should not be charged as an error for the reason that the writer made no pretension of drawing a comparison between the respective total costs including past expenditures for either railways or waterways. To have attempted that would have been futile and the results of only academic interest. Why be concerned about the economic wisdom of past expenditures, or interest charges on them? The writer's comparisons and conclusions refer only to the Mississippi Valley systems. Federal appropriations therefor were provided by taxes, not bond issues, and the public is not paying interest on them. If one must consider all this water (which is "over the dam" with respect to waterways), why not then, give similar consideration to the millions and possibly billions of dollars of capital which has been lost by investors in the re-organization of bankrupt railroads?

The writer specifically limited his discussion to future costs for railroad extensions and to the completion of Mississippi Valley waterway improvements which, at that time, had been approved and authorized by Congress.

The second error charged by Professor Kittredge is to the effect that the writer claimed as an economic advantage of the waterways an annual amount of $73 600 000 to cover interest on railway capital expenditures saved by the completion and utilization of the Mississippi Valley waterways.

The writer realized that the amount of this claimed advantage was highly questionable and so pointed out in his paper. Professor Kittredge and others called this to the attention of the writer previous to the publication of the paper. All these critics admitted there was justification for some claim of this kind, but did not suggest a method of evaluating it. After due consideration the writer decided to let the claim stand as originally made in the hope that the subject would be clarified in the discussions and that it would serve as a target to invite discussion of the entire paper, particularly by railroad officials and economists.

The Traffic World, of which Mr. Palmer is Editor and Manager, is severely critical of the development of inland waterway transportation. While undoubtedly he is well qualified, to discuss or challenge statements made by the writer and others who prepared the papers of this Symposium, Mr. Palmer has not (in so far as the writer can determine) chosen to do so. His discussion is apparently only a presentation of his personal opinion on a few of the general phases of the subject.

Mr. Palmer admits that "if it is found that water transportation fills a need that can be supplied in accordance with sound and reasonable economy it should, of course be supplied", but he refuses to accord to water transportation the right to any form of paternalism or public support during its development

period, although the preliminary development of railroads was due to this same form of paternalism.

The discussion by Mr. Brent is confined to the writer's paper and is principally devoted to showing its conservatism. This is, indeed, gratifying to the writer, particularly in view of Mr. Brent's broad knowledge of the subject gained from many years' experience with the Federal Barge Line and other transportation agencies, both water and rail.

It should be noted, however, that Mr. Brent discusses rail costs and traffic densities on the basis of miles of road, whereas the writer discusses them and presents figures on the basis of miles of track. By the latter method costs and efficiency are reduced to a basis common to all railroads, whether of single or multiple-track construction. The Pittsburgh and Lake Erie Railroad mentioned by Mr. Brent has 231.46 miles of road, but it has 1 200 miles of main track. Hence, while its road-mile traffic density (10 700 000 ton-miles) is the greatest, its track-mile density—the true measure of efficiency—is 2 060 000 ton-miles. The Virginian Railroad has the greatest track-mile density (3 400 000 ton-miles) of any railroad in the United States. Mr. Brent's statements and arguments based on miles of road are correct and would be equally applicable and forceful if reduced to a miles-of-track basis.

The railroad point of view in this controversy has been so seldom presented, except in generalities, that the discussion by Mr. Dunn, who is Editor of *The Railway Age,* stands out as a most valuable contribution to this Symposium. Mr. Dunn presents his data and arguments and conclusions in sufficient detail so as to invite specific definite argument in rebuttal. His views, however, are based upon certain fundamentals which are really the points in controversy. He denies the economic soundness of any form of public subsidy to waterways which is not fully reflected in the cost of water transportation. Recurring annual subsidies should be so reflected, but why should interest on accumulated construction costs provided by the public for channel improvement be added to operating costs when the public is not being taxed for them? There is no more reason for adding such a theoretical interest charge to waterway operation costs than there would be in adding to rail rates a fair return on the billions of dollars of capital which has been lost by investors in the re-organization of bankrupt railroads.

While Mr. Dunn separates the cost of inland waterway transportation into two parts, namely, subsidy cost and the cost of the actual movement of freight, he insists on combining the two for comparison with rail rates, in order to show that the cost of water transportation is greater than rail freight rates. It is true that the public will "foot the bill", but as National or sectional prosperity depends more on the actual moving of the freight than it does on the amount of the freight bill, it is evident that it is the freight rate which must be the governing factor, and that this rate must not be greater than the traffic will bear. For two decades the agricultural and manufacturing industries of the Middle West have been slowly "strangulating" under freight rates which the traffic could not bear, at the same time permitting the industries to prosper. The public in the Middle West at last has realized this condition and believes that in inland waterway transportation lies the solution of the

problem. Waterways are Nature's God-given free highways to Man and the laws of all great nations, including those of this country, recognize them as such. Therefore, it is an obligation of the Federal Government to maintain such of these channels as may be used to the economic benefit of the sections where they exist. The extent to which such channels will be used will depend upon the terminal and operating costs of waterway movement plus the profit required to secure the necessary capital. If the resulting freight rates attract commerce in considerable volume, the community will benefit. Some railroads may suffer for a time; but such incidents are only passing events in the march of progress, which cannot be stopped in a nation of free people. A far greater menace to present railway operations exists in the remarkable growth of truck and bus traffic. The haulage costs of truck traffic greatly exceed rail costs. This is a basic economic advantage possessed by the railroads, and hence they should be able to recapture that business.

From his viewpoint of the subsidy question Mr. Dunn correctly shows the present status of costs on the New York Barge Canal, although better authorities than the one he quotes have repeatedly asserted hidden public benefits to an amount greatly in excess of the "hidden" cost mentioned by Mr. Dunn.

The part of Mr. Dunn's discussion relative to the Ohio River system costs is answered in the discussions by Colonel Kutz and Mr. Hobbs, both of whom (as U. S. War Department officials) possessed authoritative knowledge of traffic and costs on that system.

Mr. Dunn's criticism of the writer's forecast of rail ton-mileage for 1950 is well founded. It is probably from 10 to 15% too high. This forecast (Curve *B*, Fig. 5) was derived in 1925 from railway records to 1924. Truck traffic was not considered because its importance was not realized. Authorities now state that the present truck traffic is about 10% as great as rail traffic. If this is true it would indicate that truck traffic in 1929 amounted to 45 000 000 000 ton-miles. Adding this to the rail traffic gives a combined total of 495 000 000 000 ton-miles which is exactly 1% less than the ton-miles shown for 1929 on the forecast curve, *B*, in Fig. 5 of the writer's paper. It is evident, therefore, that the forecast portion of Curve *B* should be considered as indicating the combined public demand for rail and truck traffic inclusive of such water traffic as may be diverted from those two agencies of transportation by the increase over the present volume of inland waterway transportation. Such a correction would require a downward revision in all the writer's forecasts of future railway development expenditures. This revision would not be nearly as extensive as indicated by Mr. Dunn, and to make it now would serve no useful purpose.

The writer is too firm a believer in the industrial future of the United States to accept Mr. Dunn's forecast of 575 000 000 000 ton-miles for rail traffic alone. To accept such a figure requires one to agree that the future rate of industrial progress of this nation not only will not equal the rate of the past forty years, but will progressively decline. If no greater traffic than this may be expected, then, as noted by Mr. Dunn, the writer's analysis shows that no additional trackage or equipment will be required. However, while equipment capacity has fallen off, track construction has continued, and

capital investments of the Class I roads since 1924 have increased at a rate 50% in excess of the writer's estimate for freight traffic growth.

As a closure to this discussion, attention is invited to the publication in October, 1930, by the "Bureau of Railway Economics", of a 238-page report entitled "An Economic Survey of Inland Waterway Transportation in the United States". This work contains all authoritative facts and references on the subject and their analyses from the railroad point of view, as to present and past conditions only.

Certain cost data from this report have been widely quoted as proof that the cost of water transportation is greater than existing rail rates and that, therefore, waterway improvement is unecomonic. The report contains (pages 193 to 198) a summarized review of Mississippi and Ohio River costs for the year 1928. The facts are probably correct, but if intelligently and fairly used they offer all the proof required to demonstrate the economic benefit of water transportation on the Mississippi River system. It is needless to say that the so-called subsidy items of channel maintenance and annual interest on accumulated construction capital cost for the past 100 years are included. The writer offers no protest at the inclusion of these items because the waterway traffic can stand it. The tabulations show the cost of these items per ton-mile of traffic in 1928, but the ton-mileage of that year is only a small fraction of the waterway capacity or the traffic reasonably to be expected within the next twenty years.

The ton-mile costs shown in Column (3) of Table 10 are exactly as they appear in the report (page 196) but Column (4) shows the costs based on a traffic of 45 000 000 000 ton-miles, and the total capital cost is assumed at $270 000 000 to cover the cost of completion of the project. Annual maintenance is increased to $6 000 000 for the same reason. Column (3) clearly shows the economic soundness of water transportation provided it can be developed to a satisfactory volume.

TABLE 10.—STUDY OF TON-MILE COSTS, MISSISSIPPI RIVER

Item (1)	No. (2)	1928 (3)	1938 to 1950 (4)
Annual capital cost per ton-mile	1	1.84*	0.24†
Maintenance cost per ton-mile	2	1.61‡	0.13⅔
Freight-carrying cost per ton-mile	3	4.00	4.00
Sub-total	4	7.45	4.37
Allowance of 50% for circuity	5	3.72	2.18
Total cost per ton-mile	6	11.17	6.55
Weighted average freight receipts of eight railroads near the Mississippi River	7	10.09	10.09
Total, in ton-miles	8	2 951 000 000	45 000 000 000

* 4% on $135 000 000. † 4% on $270 000 000. ‡ 4% on $4 753 000. ⅔ 4% on $6 000 000.

Referring only to the 1928 values in Table 10 the report (page 196), states:

"The rail costs average 10.09 mills per ton-mile; the estimated *minimum* transportation costs on the Mississippi River average 11.17 mills per equiva-

lent ton-mile, after allowance is made for the greater circuity of the water route."

Attention is invited to the unwarranted use of the word, "minimum", in the foregoing statement.

Inasmuch as the Bureau of Railway Economics has included in its waterway costs for 1928 all items which it considers necessary to conform to sound accounting practice, the total thus derived should be reduced by the 20% freight-rate differential saved on water shipments before a comparison is made with rail rates. The freight-carrying cost of 4 mills per ton-mile increased 50% on account of circuity or greater length of water haul over rail haul between ports becomes 6 mills. However, since the shipper, in paying this amount, saves 20% of the rail rate, his saving becomes 25% of 6 mills, or 1.5 mills per ton-mile. Subtracting 1.5 from 11.17 and 6.55, respectively, leaves 9.67 mills per ton-mile in 1928 and 5.05 mills per ton-mile by 1950, at which time (as indicated on Fig. 9) traffic will equal 45 000 000 000 ton-miles per annum on the Mississippi River alone. This indicated public cost of 5.05 mills for future traffic is the figure which should be compared with rail rates of 10.09 mills per ton-mile to determine the relative public economy of the two methods of transportation.

Thus, it is shown that, by the railroad method of accounting, waterway transportation costs to the public on the Mississippi River in 1928 were slightly less than average rail rates, and that if and when water traffic equals 45 000 000 000 ton-miles per annum, the cost will be 5 mills per ton-mile less than present average rail rates.

The Bureau report (page 198) shows, in the same way, that water costs on the Ohio River in 1928 were 12.36 mills per ton-mile and that comparable average rail, rates were 8.83 mills. The water costs covering capital and maintenance are based on 2 702 416 316 ton-miles of traffic in 1928. Fig. 9 indicates that by 1945 Ohio River traffic will be five times the present traffic, in which event the capital and maintenance charges would be one-fifth of the amounts shown by the Bureau report for 1928. By making this adjustment the indicated water cost for 1945 becomes 6.91 mills per ton-mile, or 1.92 mills less than present average rail rates.

Numerous railroad officials and economists in public addresses (many of which have been reproduced in the public press) and railway periodicals, supplemented by editorial agreement therewith, have repeatedly asserted that, if inland waterway transportation on the Mississippi River had to bear the cost of the construction and maintenance of the channel as railroads do with respect to this right of way, the cost of water transportation would be shown to be greater than the cost by rail. Their methods of accounting for capital and maintenance of channel costs are not in accord with the National policy. However, if their method is accepted as economically sound, their assertions are reasonably correct with respect to present-day traffic, but very incorrect if water traffic develops in large volume. Economic justification, therefore, depends not upon present-day costs, but upon the development of a large volume of water traffic which will reduce these overhead, or so-called subsidy, items to almost negligible figures.

E. A. HADLEY,[8] M. Am. Soc. C. E. (by letter).[8a]—The several discussions of the writer's paper have been given careful consideration. The operating statistics and capital expenditures made by the railroads are readily available to any one interested in the subject, from the reports which are required to be made to the Interstate Commerce Commission; and all such statistics and reports are made on a uniform basis in a manner prescribed by the Commission.

On the other hand, information and statistics regarding waterway development and operation must be gathered with great effort from a large number of official reports of various departments of the Government. Many of these reports are issued by the Chief of Engineers of the United States Army and others appear in the reports of the Inland Waterways Corporation, a Government organization. The annual reports of the Chief of Engineers consist of bound volumes aggregating from 3 000 to 6 000 pages, from which it is no small task to assemble data pertinent to the subject under discussion.

The Bureau of Railway Economics, which has been established by the railways of the United States for the scientific study of transportation problems, issued a volume, designated as Special Series No. 56, and dated Washington, D. C., October 15, 1930, entitled: "An Economic Survey of Inland Waterway Transportation in the United States." The information and statistics given in the volume are as nearly correct as it is possible to have them, based upon the official and unofficial sources of information from which it has been compiled. This volume furnished a considerable portion of the foundation for the discussion in the writer's paper.

A joint report[9] of the U. S. Corps of Engineers and the United States Shipping Board issued in 1927 has laid down the principle that when the cost of water transportation is to be compared with the corresponding cost of rail transportation, the former should include interest charges on the capital investment in the waterway improvements plus the annual cost of operation and maintenance. Using these basic principles, and applying them to available statistics regarding the Mississippi River, the Ohio River, and the New York State Barge Canal, the following situation is developed:

(1) Transportation costs on the Mississippi River are not less than 11.17 mills per ton-mile, with allowance made for circuitous river channels; on eight railroads which, in general, parallel that river, the average inclusive freight charge per ton-mile is 10.09 mills.

(2) Transportation costs on the Ohio River, determined in a similar manner, are not less than 12.36 mills per ton-mile; on seven railways which, in general, parallel that river, the average inclusive freight charge per ton-mile is 8.83 mills.

(3) Transportation costs on the New York Barge Canal are computed at 19.41 mills per ton-mile; on the railways of the Eastern District the average inclusive freight charge per ton-mile is 10.90 mills.

It must be remembered that the financial statements issued by the Inland Waterways Corporation do not make any allowance for return on the capital stock issued by it to the Government. The expenses of the Corporation do

[8] Chf. Engr., Mo. Pac. R. R., St. Louis, Mo.

[8a] Received by the Secretary, February 16, 1931.

[9] "Transportation on the Ohio River System" (Interim Rept.), 1927, p. 17.

not include any provision for taxes because the Government pays no taxes to itself. These items must be considered a charge against revenue before any private investor would judge the operations of the Barge Line to have carried out the purpose for which it was established, namely, to demonstrate whether or not the service can be carried on by private capital at a profit.

Because of lack of statistics it is not possible to compare the operations of private river carriers and those of the Government Barge Line with respect to their profitableness. Even if the private carriers use free-of-charge, improved waterways made possible by expenditures of Government funds, they do pay taxes and may be presumed to earn a return on their investment or they could not remain in business. The same condition is true of those companies which utilize the improved waterways for hauling their own raw materials and products in that they use the waterways free of tolls and find the water transportation a more or less profitable extension of their plant operations.

The statement is repeatedly made that the present means of transportation will not suffice to meet the future growth of traffic incident to the growth of the country in population and wealth. President Roosevelt in his letter creating the Inland Waterways Commission, in 1907, wrote:[10]

"It is common knowledge that the railroads of the United States are no longer able to move crops and manufactures rapidly enough to secure the prompt transaction of the business of the nation, and there is small prospect of immediate relief. * * * There is reason to doubt whether any development of the railroads possible in the near future will suffice to keep transportation abreast of production."

President Roosevelt had before him the relatively large rail-traffic totals for 1906 when he wrote this letter. Despite his predictions, however, the rail traffic of 1929 was more than double that of 1906, and this increased traffic was moved promptly without congestion and without delay to shippers or receivers.

In its 1908 Annual Report, the Inland Waterways Commission, stated:[11] "It is clear that at seasons recurring with increasing frequency they (the railways) are unable to keep pace with production or to meet the requirements of transportation."

The National Waterways Commission in 1912 stated[12] that the rapid expansion of traffic between 1904 and 1907 had exhausted the surplus capacity of the railroads and added: "With the next period of expansion there is strong probability of a worse congestion of traffic than occurred before."

While he was Secretary of Commerce, President Hoover made addresses throughout the country advocating the consolidation of the scattered sections of improved waterways. In this connection he made the statement that the growth of population within twenty-five years would produce such an increase of traffic that the railways would be unable to meet it. Using past history as a basis for forecast of the future, the railways will undoubtedly keep ahead

[10] Preliminary Rept., Inland Waterways Comm., 1908, p. 16.
[11] *Loc. cit.*, 1908, p. 19.
[12] Final Rept., National Waterways Comm., 1912, p. 575.

of the growing demands of traffic upon them. For several years no traffic congestion has occurred and reports of the Department of Commerce have emphasized the remarkable improvement of the transportation service rendered by the railway industry.

The construction and operation of any system of transportation involves large expenditures of money and it may be briefly considered that the principal items of expenditure are, as follows:

(1) The cost of construction of the system itself, which usually runs into hundreds of millions of dollars for any large project. In the case of a railroad its cost covers the right of way, tracks, and other permanent structures. In the case of an inland waterway, it covers the cost of construction of the canal or improvement of the river, with locks, dams, and other appurtenant structures.

(2) The annual charge for capital; that is, the annual interest charge on the money invested in the construction or improvement.

(3) The cost of necessary equipment, whether in the form of railroad locomotives and rolling stock, or of ships and barges.

(4) The annual carrying charge for capital invested in equipment.

(5) The cost of maintenance, both of the transportation route (railroad right of way and tracks, or river and canal), and of the equipment operated thereon.

(6) Proper charges for depreciation on plant and equipment.

(7) Necessary payments of insurance premiums and taxes.

(8) The direct cost of the transportation service itself, including wages, cost of fuel, materials, and supplies, and the cost of supervision and general expense.

Unless all these items are considered and accounted for, no fair comparison of the relative costs of the two forms of transportation can be made.

The paper presented by Mr. Cornish as a part of the Symposium makes an attempt at forecasting the annual freight transportation demand of the United States by 1950, together with an estimate of additional railroad facilities required to provide for proper handling of that portion of the increase which would go to the railroads.

In discussion, Professor Sorrell sets forth in a most convincing manner the fallacy of attempting to forecast the probable freight traffic to be carried by the railroads in 1950. Furthermore, he states the results of similar forecasts brought down to the year 1943 showing such wide variances that he reaches the conclusion that "it is unsafe to assume that the trend for the next twenty-five years will be on the same basis as that for the last forty."

In his comparison of costs of transportation between the railroads and the waterways, Mr. Cornish overlooks the fact that the present average ton-mile earnings of the railroad include return on the investment and taxes, and he has also used the arbitrary differential of 20% in water rates under the rail rates as his measure of the saving in transportation costs by use of the waterways.

Mr. Cornish has further disregarded capital outlays already made for river improvements and terminals by use of public funds, except those terminals used by the Federal Barge Line. He also assumes that the present average freight receipts of the railroads of about 11 mills per ton-mile will prevail in

1950; however, should the increase in traffic on the railroads develop to the point estimated by him, it is probable that freight rates could be reduced materially, which would lessen the savings he estimates due to waterway improvements.

On the other hand, if all the figures going to make up the cost of waterways transportation are considered, the estimated savings on this account would largely disappear, particularly if the gain in waterways traffic did not reach the total of his estimate.

Many of the same fallacies in reasoning appear in the discussions by Colonel Kutz, and Mr. Hobbs. Mr. Brent comments somewhat on the paper by Mr. Cornish and toward the end of his discussion makes the following statement:

"The day of cheap railroad transportation is past. Sweeping increases in rates are being made in all important territories to bring railroad revenues to the mark fixed upon as necessary to support and expand the plant."

In 1922, the Interstate Commerce Commission ordered a 10% general reduction of freight rates in the belief that this would give "fuller assurance to the carriers of realizing the fair return contemplated by the law." Since that date the Commission has granted no "sweeping increases in rates", but on the contrary there has been a continual paring downward of rates so that the present per ton-mile earnings of the railroads are less than they were following the reduction made in 1922, and there appears to be no indication on the part of the Commission of allowing any general increase in rates.

In the last analysis it appears that most of the proponents of waterways transportation give consideration to only a part of the costs of such service in making their comparisons, the remainder being borne by the Federal Government, or other public authority, while the railroad figures include all the elements of such costs and are borne directly by the railroads themselves. The total cost of each form of transportation should constitute the only basis for comparison.

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DISCUSSIONS

THE TRAINING WALL ACROSS THE LIAO BAR IN MANCHURIA

Discussion

BY P. N. FAWCETT, M. AM. SOC. C. E.

P. N. FAWCETT,[1] M. AM. SOC. C. E. (by letter).[1a]—Mr. Goodrich has supplemented the paper considerably with particulars of the ports of North China and their respective advantages and disadvantages. His previous association with the Chihli River Commission makes his remarks of particular value. As he states, all the rivers of North China are obstructed by ocean bars. These bars and the very extensive mud-flats, which reach many miles out to sea beyond the coast line, are formed from the deposits of silt brought down from up river during the "freshet" season. Mr. Goodrich states that the percentage of silt in the waters of the Hai River on occasions has been as high as 10%, furthermore, the silt is composed of the so-called "loess" soil and is of so fine a nature that it offers considerable difficulty to being removed by drag-suction, hopper dredges because its fineness prevents its settlement in the hoppers.

This class of dredge has been used on the Hai River Bar since about 1914, but the plant is insufficient to obtain a depth of more than 9 ft. at low water, and the maintenance of this depth offers considerable difficulties, so that it is frequently reduced to 6 or 7 ft. during the "freshet" season.

As Mr. Goodrich states, in 1925, the construction of two training walls was undertaken, at the mouth of the Hai River. These walls were completed in 1927, and the work of dredging a channel between them was commenced at the opening of the 1928 season, by means of large bucket dredges. The walls were built to a height of high-water neap tides and extended seaward across the south flat, but only to a point where a low-water depth of 4 ft. naturally obtained. The sites of the two walls at the bend in the Hai River known as Deep Hole, are shown on Fig. 11. They were to extend about 3.5 miles into the sea beyond Point *C*.

NOTE.—Author's closure. The paper by P. N. Fawcett, M. Am. Soc. C. E., was published in May, 1930, *Proceedings*. Discussions of the paper have appeared in *Proceedings*, as follows: September, 1930, by R. D. Goodrich, M. Am. Soc. C. E.; October, 1930, by Messrs. Herbert Chatley, and H. C. Ripley; and December, 1930, by B. Okazoki, M. Am. Soc. C. E.

[1] Engr.-in-Chf., Lower Liao River Conservancy, Newchwang, Manchuria, North China.

[1a] Received by the Secretary, January 28, 1931.

Unfortunately, when dredging was begun, no sill was placed at *Q C* across the Deep Hole where there is a low-water depth of 40 ft., and to make matters worse, the engineers placed a dam across the entrance between the two dikes. This dam was carried to the same height as the dikes, and, therefore, it placed the dredges in a kind of dock. It is evident that if any success was to be obtained, no dam should have been placed across the entrance, but on the other hand, every inducement should have been offered to the ebb currents to enter the cut and assist the dredging. A submerged dam, built to 10 ft. below low water (so as to allow navigation with the same depth as was available across the bar) should have been placed across the Deep Hole at *Q C* (Fig 11), thus throwing as much of the current as possible between the dikes. The dikes should have been extended into deep water and the use of timber should have been avoided.

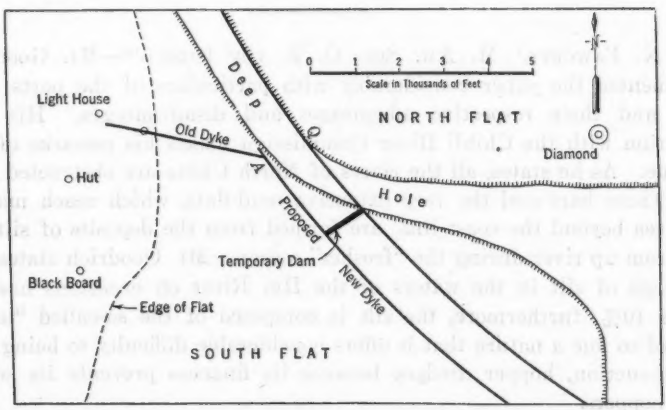

FIG. 11.—SYSTEMATIZATION OF THE HAI RIVER ENTRANCE AND BAR CHANNEL

It was found that after two years intensive dredging, the mud level still remained the same between the dikes as it was outside them, because these methods had not been adopted.

The Conservancy Board, in 1929, called in a Commission of Consulting Engineers, of which the writer was a member, to report upon the project. This Commission—partly because the financial plan resources of the Conservancy were insufficient to pursue the plan further, and partly from a technical viewpoint—recommended the abandonment of the plan and the pulling up of the dikes.

In the writer's opinion, the dikes were incorrectly located, the present natural Deep Hole Channel should have been utilized and trained and carried out in its continuation across the bar. As it is, more than $1 000 000 (gold) was spent on this plan, all of which was a total loss.

Mr. Goodrich has always felt that a single curved jetty across the Hai Bar might have provided the desired benefits. The writer concurs with this opinion, and such a wall was designed and proposed by Mr. A. de Linde, as

long ago as 1902, while he was Engineer-in-Chief of the Hai River Commission.

Dr. Chatley gives some interesting data of the Whangpoo River and describes results obtained by the single training wall originally constructed at its mouth, which was afterward supplemented by an additional wall. In this particular locality, the single wall increased the depth across the bar from 15 to 21 ft., and the double wall further increased this depth to 30½ ft. He is of the opinion that the results obtained on the Whangpoo River show that the scour induced by a single training wall can be usefully supplemented by a double wall.

As Dr. Chatley states, the conditions in the two localities are very different. In the case of the Whangpoo the bar was situated where the river debouched into the Yangtze River, a fairly sheltered locality as compared to the sea coast. Every situation must be judged on its merits and each of them presents its own particular problems. In the case of the Liao Bar, where the ebb and flow of the tidal currents take place in the directions described in the paper, and where the bar is situated 8 miles from the river entrance, the writer is of the opinion that the construction of a second training wall would necessitate too great a risk, that the expenditure involved would not be justified by the increase in the depth that might or might not result. Added to this, and partly owing to the intensive ice action, a very high annual maintenance charge would need to be added to the capital cost. It was with these points in view that the writer limited the construction to a single wall.

Mr. Ripley[2] considers that the increase in depth across the Liao Bar is caused by overfall of the flood tide. This, however, is not the case, there being practically no overfall, the flow of the flood tide taking place very quietly and the surface of the water being at the same level on both sides of the wall. The writer is, of course, familiar with Mr. Ripley's proposed method of deepening a bar by beginning the construction from the seaward end, and working inshore, but considers that it would be difficult to persuade commercial bodies to sanction such a method of work.

Dr. Okazaki is quite correct in pointing out the difficulties of improving the flow of the upper river owing to the refusal of the Chinese authorities to take any constructive interest in matters of conservancy. Had funds been forthcoming, the training wall across the Liao Bar could have been completed in two instead of ten years. Furthermore, it is probable that a greater improved depth would have been obtained, since the alteration of the surface slopes would have taken place more suddenly.

The writer agrees with Dr. Okazaki that the tidal flow is of the greatest importance in obtaining and maintaining a channel across the bar. He does not concur, however, with the view expressed, that the improvement of the fresh-water flow should not be undertaken pending the completion of works on the bar. Far from this, the records of this locality show that the deterioration of the bar dated from about the time when the Liao River was diverted

[2] Mr. Ripley died on January 9, 1931.

down the Shuangtaitzu River. Furthermore, since that date the entire estuary of the Liao has become encumbered with sand banks and islands which are accreting with great rapidity and thus excluding the tidal influx, so that it is only a matter of time—unless adequate steps are taken—when the estuary will be filled up.

It is a well-known axiom that "tidal rivers cannot be maintained without their fresh-water flow." It was with the intention of securing its original fresh-water discharge back to the Liao Estuary, that works were built by the Conservancy Board in the upper river and the weir and cutting described by Dr. Okazaki were constructed at a cost of approximately $700 000 (gold).

In spite of the writer's continued warnings, these works are not being utilized to their full extent, and practically no reversion of fresh water has occurred since they were completed. Unfortunately, the Board has no control over the area, as it is under Chinese jurisdiction. In the case in point, instead of causing the cutting to be scoured to its full width of 800 ft. and the fresh water reverted past Newchwang, the cut is maintained at a width of about 100 ft., and the remainder of the land between the embankments is leased by the officials for farming in spite of the fact that this land was bought by the Conservancy Board.

Works built up country in China are also subject to much damage and interruption by the local population and banditti, and a weir constructed across the Shuangtaitzu River in 1912 to re-divert the flow back into the Liao was destroyed by the villagers as soon as it was completed, because it interfered with the lucrative practice of salt smuggling.

Dr. Okazaki believes that it is better for the estuary and bar if this fresh-water discharge is allowed to continue to escape down the Shuangtaitzu River as it carries an extremely high silt content, whereas the tributaries, the Taitzu and the Hun, are comparatively siltless. The writer's views are entirely the opposite, and he would point out that the silt of the Shuangtaitzu is of so extremely fine a character that it is all carried completely across the bar into deep water at sea before settlement occurs. The Hun River and Taitzu River, on the other hand, show clean and clear fresh-water discharges; yet, in time of heavy rains, these two rivers roll large quantities of extremely heavy sand along the bottom, which, on one occasion, completely silted the Newchwang Estuary, to a depth of 12 ft.

During this "freshet" samples taken for silt content, 1 ft. above the river bed, gave no indication of this sand. In view of this, the silt table (Table 1) presented by Dr. Okazaki would give no indication of the true nature of the problem.

The fineness of the Upper Liao silt is somewhat remarkable as shown in Table 2. In this study, 51% of the Liao soil was found to have a grain diameter of less than 0.050 mm.

In writing this paper to describe the effect on the Liao Bar of constructing the single training wall, the writer felt that particular interest would attach to the effects, considering the fact that no dredge had been used in collaboration with the construction of the training wall. The reason for this

was entirely due to the fact that funds available were not sufficient for its purchase.

In 1929, the author was enabled to purchase the drag-suction hopper dredge, *Java,* from the Dutch East Indies Government. This vessel, built in 1911, is fitted with twin pumps and propellers driven by engines of 560 and 900 i.h.p., respectively. The suction pipe is 37½ in. in diameter and the hopper of 750 cu. yd. capacity.

TABLE 2.—COMPARISON OF FINENESS OF SILT

Item	Ymuiden	Seine	Rotterdam Waterway	Hai-Ho	Liao Ho
Percentage passed through sieve of 4 900 meshes per square centimeter.....	2	23	45	48	94
Mean size of grain, in millimeters*.....	0.167	....	0.102	....	0.053

* After method described by Messrs. T. Heyblom and A. G. Maris, Paper 76, XIV Inter. Cong. of Navigation.

Dredging was begun in April, 1930, and in spite of many interruptions, due to the enormous quantities of iron bars, wire rope, Chinese anchors, etc., that kept interrupting the work, a channel 12 ft. deep and 200 ft. wide had almost been completed by the end of September. Freshets then brought down large quantities of sand from the Taitzu and Hun Rivers, with the result that the dredged cut almost became obliterated.

With the Shuangtaitzu fresh-water discharge returned to the Liao Estuary the writer is of the opinion that the damaging sand deposits of the Taitzu and Hun Rivers would be flushed across the bar and deposited out at sea. The situation is very similar to that at Tientsin to which Mr. Goodrich calls attention, where disaster is caused to the improvement works in the lower river and harbor, by the neglect that has been allowed, of the upper portions of the rivers and tributaries. The writer is, therefore, not in agreement with Dr. Okazaki and considers that the permanent improvement of the bar is impossible, unless the tributaries and upper river are first put in order.

As regards the dredging plant required, a 14-ft. channel across the bar should be obtainable with the dredge, *Java,* as long as no further excessive sand deposits occur.

The writer does not consider that the height of the waves ever exceed 10 ft., and no flattening of the channel has ever been experienced after storms. On the other hand, at the Hai River Bar, the channel is beaten down and reduced in depth after each severe storm. The failure of such storms to flatten the Liao Bar, the writer believes, is entirely due to the protection offered to the channel by the training wall, which receives the force of the waves on its convex side.

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DISCUSSIONS

STRESSES IN GRAVITY DAMS BY PRINCIPLE OF LEAST WORK

Discussion

BY MESSRS. FREDRIK VOGT, AND EUGENE KALMAN

FREDRIK VOGT,[17] ASSOC. M. AM. SOC. C. E. (by letter).[17a]—Gravity dams are now designed to heights which surpass those of all previous dams, and hence stresses are approaching the allowable limit. For such large dams an eventual saving by reducing the ratio, base width to height, is so great that it pays to improve the analysis in order to find the lower limit of this ratio which can be used without danger, and to ascertain the actual value of the safety factor. It is fortunate, therefore, that the author has chosen for discussion the stress problems for gravity dams and has thus broadened the knowledge of engineers on this subject.

The principal question is whether or not the classical theory, which supposes a linear distribution of the vertical normal stresses, is on the safe side. The tensile stresses are the most dangerous, particularly those at the up-stream face, since the tensile strength of concrete is not to be trusted, and since horizontal tensile cracks at the up-stream face will eventually cause uplift forces from water penetrating into such cracks. The greatest emphasis, therefore, should be laid on a correct computation of the stresses at the up-stream face, and the design should be executed so that all possibilities of vertical tensile stresses at the up-stream face are eliminated as far as possible for all dams in which stability depends mostly on gravity action.

According to the classical theory, the triangular-shaped dam in Fig. 1 will be exposed to maximum principal stresses, $n_1 = -g\,y\left(1 + \frac{1}{k^2}\right)$, at the down-stream face when loaded, and $n_2 = -w\,y$, at the up-stream face when unloaded. Smaller ratios of base width to height, than $k = \sqrt{\frac{g}{w}}$ are never used. With $g = 62.5$ and $w = 150$ lb. per cu. ft., this gives $k = 0.645$, and a maximum

NOTE.—The paper by B. F. Jakobsen, M. Am. Soc. C. E., was published in September, 1930, *Proceedings*. Discussion of the paper has appeared in *Proceedings*, as follows: September, 1930, by the late William Cain, M. Am. Soc. C. E.; November, 1930, by Messrs. William Gore, L. J. Mensch, Johannes Skytte, and Lars R. Jorgensen; January, 1931, by Messrs. C. A. P. Turner, Donald P. Barnes, and Anthony Hoadley; and February, 1931, by Messrs. Fred. W. Ely, and M. H. Gerry, Jr.

[17] Docent, Norges Tekniske Höiskole, Trondhjem, Norway.

[17a] Received by the Secretary, January 21, 1931.

compression of 600 lb. per sq. in. is then obtained for $y = 407$ ft. Even if the classical theory is not accurate, this will suffice to prove that the magnitude of the compressive stresses in gravity dams has no direct significance in the design before the height is approaching or surpassing 400 ft., assuming that first-class concrete and rock foundation are obtained.

It seems desirable to study in detail the solution by the classical theory for comparison with the author's method. Using the notations from Fig. 1 the author gives the stresses according to the classical theory in Equations (13), and these stresses satisfy the equilibrium equations and the boundary conditions both at the up-stream and the down-stream faces. Let e_x and e_y be the displacements of a point, x, y. For the two-dimensioned system the compatibility equations are, therefore,

$$E \frac{\delta e_x}{\delta x} = n_x - \frac{n_y}{m} \quad \ldots\ldots(63)$$

$$E \frac{\delta e_y}{\delta y} = n_y - \frac{n_x}{m} \quad \ldots\ldots(64)$$

and,

$$t = G \left(\frac{\delta e_x}{\delta y} + \frac{\delta e_y}{\delta x} \right) \quad \ldots\ldots(65)$$

in which, $m E = 2 (m + 1) G$.

The displacements, e_x and e_y, can be eliminated after derivation of Equation (63) for e_x twice with respect to y, of Equation (64) twice with respect to x, and of Equation (65) twice with respect to x and y. By introducing the equilibrium Equations (6) and (7), t and the derivative of n_x with regard to x can also be eliminated, and the following compatibility equation is obtained:

$$\frac{\delta^2 n_y}{\delta x^2} + 2 \frac{\delta^2 n_y}{\delta y^2} + \frac{\delta^2 n_x}{\delta y^2} = 0 \quad \ldots\ldots(66)$$

By a repeated double derivation n_x also can be eliminated, but the form of Equation (66) is convenient for the purpose in question.

The stresses, Equations (13), satisfy the compatibility equation, and the displacements, e_x and e_y, can be found, therefore, by integration:

$$e_x = \frac{1}{E} \left[\left(\frac{g}{k^3} - \frac{w}{2k} \right) y^2 - \left(g + \frac{g}{m k^2} - \frac{w}{m} \right) x y + \left(\frac{g}{m k^3} - \frac{w}{2 m k} \right) x^2 - a y + c \right] \quad \ldots\ldots(67)$$

and,

$$e_y = \frac{1}{E} \left[\left(\frac{g}{2m} + \frac{g}{2k^2} - \frac{w}{2} \right) y^2 - \left(\frac{2g}{k^3} - \frac{w}{k} \right) x y - \left(\frac{g}{k^2} + \frac{g}{2 m k^2} - \frac{g}{2} + \frac{w}{2m} \right) x^2 + a x + d \right] \quad \ldots\ldots(68)$$

The coefficients, a, c, and d, give only the position of the deformed section relative to the co-ordinate system, but have no influence on the stresses or on the deformation of the section.

The dead load is placed successively during the construction period, and part of the dead load deformation, therefore, is obtained before the section is

completed. Equation (68) is, therefore, not valid for the dead load. The fresh concrete also will be partly plastic and not elastic; and the modulus of elasticity is lower for fresh than for aged concrete. Hence, it seems better to consider the stresses and deformations due to water load only, which are found by making w equal to zero in the equations. The author's criticism of the classical theory also mainly concerns the stresses due to water load only.

For the foundation line of the section, y $(= h)$ is a constant, and the vertical displacements are then given by a parabolic curve convex downward since the equation for e_y contains a term, x^2, with a negative coefficient. Assuming that $m = 6$ and $k = 0.645$ the rise of the curve is found to be,

$$\frac{1}{4}\left(1 + \frac{1}{2\,m} - \frac{k^2}{2}\right)\frac{g\,h^2}{E} = 0.22\,\frac{g\,h^2}{E} \quad \ldots\ldots\ldots\ldots (69)$$

At the same time, the width of the base line is reduced with $k\,\frac{g\,h^2}{E}$ $= 0.645\,\frac{g\,h^2}{E}$.

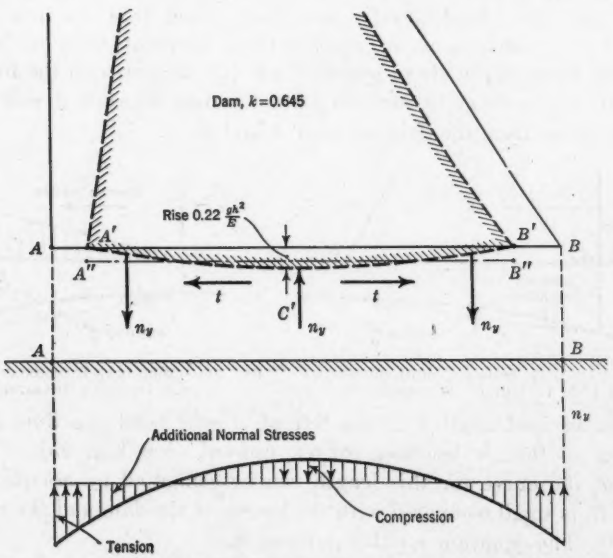

FIG. 24.—DEFORMATION OF DAM SECTION DUE TO WATER LOAD ONLY, ACCORDING TO CLASSICAL THEORY, AND ADDITIONAL STRESSES TO BE APPLIED IN ORDER TO MAKE DAM SECTION FIT AN UNYIELDING FOUNDATION

If the foundation is unyielding, additional stresses would have to be applied, as shown in Fig. 24, in order to straighten the line, $A'\,C'\,B'$, and to lengthen it to fit the line, $A\,B$, of the foundation. Shearing stresses directed from the center toward A and B are necessary in order to increase the length, but these acting alone would increase the rise. Hence, the additional normal stresses must be tensile stresses near both faces and compressive stresses at the center; that is, the corrected normal stresses due to water load only must be tensile stresses, greater than those found by the classical theory near the up-

stream face, and compressive stresses, less than those found by this theory at the down-stream face. The tendency of these corrections is the same as that found by the author (Fig. 6).

The stresses represented by Equations (13) resulting from the water load only, will lengthen the height, h, of the up-stream face with $\frac{g\,h^2}{2\,k^2\,E} = 1.2\,\frac{g\,h^2}{E}$. To straighten the line, $A'\,C'\,B'$, so that it coincides with the line, $A''\,B''$, when the distance, $A'\,A''$, equals $\frac{2}{3}$ times $0.22\,\frac{g\,h^2}{E}$, an average change in stresses at both faces equal to $\frac{2}{3}$ times $\frac{0.22}{1.2}$, or 12%, approximately, is required. Because of uneven distribution of these additional stresses over the height, and because of additional curving due to shearing stresses, the corrections to be added may be somewhat larger at the base.

The foregoing remarks are based on the assumption that the foundation is unyielding, but it will always yield elastically, and the deformations can be computed for any given load on a plane foundation, as shown in a paper by the writer.[18] It is convenient to consider three different types of loads on the foundation, namely, the direct pressure, $g\,h$, per unit area on the foundation at the left of A, the shearing stresses from the dam between A and B, and the bending stresses from the dam between A and B.

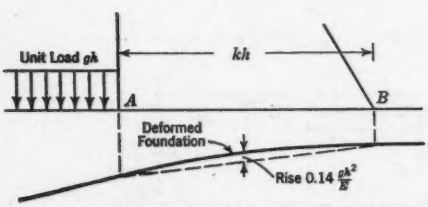

FIG. 25.—VERTICAL DEFORMATION OF FOUNDATION DUE TO DIRECT WATER LOAD

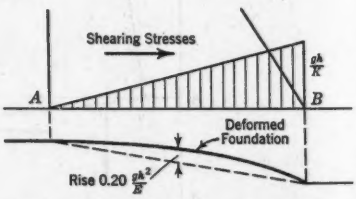

FIG. 26.—VERTICAL DEFORMATION OF FOUNDATION DUE TO SHEARING STRESS

The direct load applied at the left of A will bend the line, $A\,B$, of the foundation so that it becomes convex upward (see Fig. 25). The middle ordinate of the curve for this length can be computed by assuming that the length, $A\,B$, is small compared with the length of the dam and the width of the reservoir.[19] The equation for this purpose is,

$$\frac{m^2-1}{m^2} \times \frac{1\,n\,2}{\pi} \times k\,\frac{g\,h^2}{E_r} = 0.14\,\frac{g\,h^2}{E_r} \quad \ldots\ldots\ldots\ldots (70)$$

in which, E_r is the modulus of elasticity for the rock, and m and k are introduced with the values used before.

The shearing stresses, with distribution, as given by Equations (13), will also curve the foundation line, $A\,B$, the same way (see Fig. 26). The middle ordinate in this case is computed by means of the general formula:[20]

$$\frac{(m-2)\,(m+1)}{4\,m^2}\,\frac{g\,h^2}{E_r} = 0.20\,\frac{g\,h^2}{E_r} \quad \ldots\ldots\ldots\ldots (71)$$

[18] "Ueber die Berechnung der Fundamentdeformation," Det Norske Videnskaps-Akademi, Oslo, 1925.

[19] *Loc. cit.*, p. 7.

[20] *Loc. cit.*, p. 26.

Both these curvings of the foundation line increase the disagreements already shown in Fig. 24. The total rise of the different curves is (assuming that $E_r = E$), $(0.22 + 0.14 + 0.20)\frac{g\,h^2}{E} = 0.56\frac{g\,h^2}{E}$. This must be compensated by stresses, as shown in Fig. 24, which will deform both dam and foundation. Since the foundation is assumed to yield, the method of estimating used in connection with Fig. 24 cannot be applied directly to this case. The writer believes that the stresses due to water load may be changed as much as 25 to 30% because of this curving. According to Table 5 the author found an increase by his method of +37% and −23% at the two faces, which checks very well with the writer's estimate.

The bending stresses on the foundation due to water load on the dam are, according to the classical theory, $\pm\frac{g\,h}{k^2}$ at both faces (see Fig. 27). These stresses will deform the foundation to an **S**-shape, as shown in Fig. 27. The average tilting is represented by a line, $A'\,B'$, or,

$$A\,A' = B\,B' = \frac{m^2-1}{m^2}\,\frac{3}{2\,\pi\,k}\,\frac{g\,h^2}{E_r} = 0.72\,\frac{g\,h^2}{E_r} \quad \ldots\ldots\ldots\ldots (72)$$

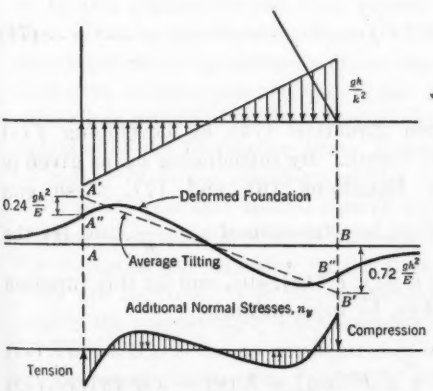

FIG. 27.—VERTICAL DEFORMATION OF FOUNDATION DUE TO MOMENT STRESSES AND ADDITIONAL NORMAL STRESSES

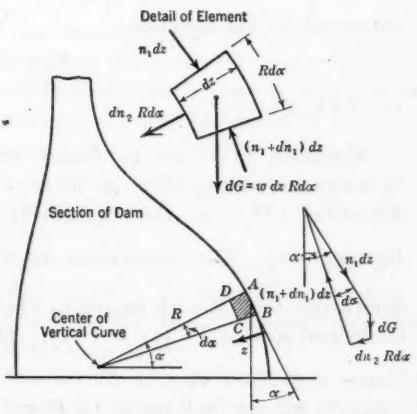

FIG. 28.—STRESSES AT CONVEX CURVED DOWN-STREAM FACE

The maximum deviation from the straight line, $A'\,B'$, is found at A and B by the formula,

$$A'\,A'' = B'\,B'' = \frac{A\,A'}{3} = 0.24\,\frac{g\,h^2}{E_r} \quad \ldots\ldots\ldots\ldots (73)$$

which in this case is equal to 20% of the total tensile deformation at the upstream face. In order to adjust the dam and foundation so that they fit together, additional stresses must be applied as shown in Fig. 27. It is difficult to estimate the magnitude of these stresses. However, since they change from maximum tension to maximum compression in a much shorter length than the additional stresses shown in Fig. 24, it is obvious that they must also

decrease to zero at depths which are correspondingly smaller. In order to obtain a large deformation in a short distance, the additional stresses (Fig. 27), must be rather large, but the effect will be negligible at a comparatively small distance from the foundation joint.

The base width, $A\,B$, will be changed by the deformation of the foundation, which again may cause some new redistribution of the stresses.

All the foregoing computations concern the stresses due to water load only. If the dead load stresses were applied after the construction was completed, the total difference between the deformed dam and foundation was reduced to about one-half. Furthermore, the additional stresses due to these disagreements then would be reduced to one-half. However, for reasons previously discussed, such computations for the dead load deformations cannot be considered to be correct.

The author's method is based on the assumption that the stresses, n_y, are distributed in the same manner for two joints at different elevations and that the magnitude at two points similarly located in these joints is proportional to y. Two points are similarly located in the horizontal joints if the ratio, $z = \frac{x}{y}$, is the same for both points, and this assumption, therefore, can be expressed by the formula,

$$n_y = y\,f\,(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

in which, $z = \frac{x}{y}$.

Equation (15) can be found from Equation (74) by developing $f\,(z)$ in series, neglecting all terms after the fourth. By introducing n_y, as given in Equation (74), in the equilibrium Equations (6) and (7), these can be integrated. For convenience, let $F\,(z)$ be a function of $z = \frac{x}{y}$, and let the derivatives thereof with regard to z be $F'\,(z)$, $F''\,(z)$, etc., and let this function be defined so that $F''\,(z) = f\,(z)$. Then, by integration,

$$n_y = y\,f\,(z) = y\,F''\,(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

$$n_x = y\,[6\,F\,(z) - 4\,z\,F'\,(z) + z^2\,F''\,(z)] + h\,(y) - x\,g'\,(y) \ldots\ldots (76)$$

and,

$$t = y\,[-\,w\,z - 2\,F'\,(z) + z\,F''\,(z)] + g\,(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (77)$$

in which, $g\,(y)$ and $h\,(y)$ are two independent functions of y. The correctness of Equations (75), (76), and (77) can easily be checked by introducing in Equations (6) and (7), which are satisfied since $\frac{\delta\,z}{\delta\,x} = \frac{1}{y}$ and $\frac{\delta\,z}{\delta\,y} = -\frac{z}{y}$.

At the down-stream face, $z = k$, and for this value of z Equations (75) to (77) must satisfy the boundary conditions defined by Equations (8) and (9), whereby the functions, $g\,(y)$ and $h\,(y)$, are restricted to the forms,

$$g\,(y) = a\,y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

and,

$$h\,(y) = b\,y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

in which, a and b are constants. After introducing these functions in Equations (75) to (77), both t and n_x can be expressed by y times the functions of z only; that is, by equations of the type given in Equation (74).

The stresses must satisfy the compatibility Equation (66), and by introducing the stresses defined by Equations (75) to (79) in Equation (66), the latter is reduced to,

$$\frac{(z^2+1)^2 F'''' (z)}{y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (80)$$

Since the squared polynom cannot become zero, the compatibility equation is reduced to,

$$F'''' (z) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (81)$$

which, integrated twice, gives

$$F'' (z) = A + B z \ldots\ldots\ldots\ldots\ldots\ldots\ldots (82)$$

in which, A and B are constants, and, hence,

$$n_y = y F' (z) = y \left(A + B \frac{x}{y} \right) = A y + B x \ldots\ldots\ldots (83)$$

This equation is identical with Equation (10) representing the classical theory.

In this manner it has been proved that no stress functions exist, except those given by the classical theory, which satisfy the condition of equilibrium, the compatibility equations, and the boundary conditions, and which at the same time give stresses proportional to the depth, y, at the joint considered. The writer has proved that the classical theory does not satisfy the compatibility equation at the foundation line. Hence, no exact solution can be found at all by means of stress functions of the type of Equation (74), or by Equation (15), which is only a special case of Equation (74).

The classical theory gives stresses which are greatly in disagreement with the compatibility equation at the foundation joint, and several large corrections are required. For non-triangular sections the classical theory does not satisfy the compatibility equation even in the dam proper.

The author's theory gives stresses which include approximately a number of necessary corrections, namely, all except those due to the S-shaped deformation of the foundation. It has been proved that the method can be only approximate since the compatibility equation is not satisfied. An exact solution can be obtained only by stress functions of a more complicated type. However, even an approximate solution means a great step forward.

The deformation of the foundation is the key of the correct solution since a change in this deformation will also change the stresses in the dam proper. Exact solutions including this point are difficult to find, particularly because of the corners at the heel and the toe of the dam, which represent singular points.

The best check on the theory can be obtained by model tests. The English tests indicate large tensile stresses at the heel of the dam in agreement with the author's theory and with the additional stresses shown in Fig. 27, but in disagreement with the classical theory. Furthermore, at some distance from

the base, these tests give a slight curving of the diagram for the vertical stresses in agreement with the author's theory. The writer has been connected with tests on models of arch dams, and such a curving of the stress diagrams was found probable. However, these tests were conducted with other problems in view, and the results regarding this particular problem are rather indirect and should not be taken as a definite proof. It is to be hoped that the model tests demanded by the Consulting Board for the Hoover Dam may give final solutions of some of these problems.

Tensile stresses may also be produced by volumetric changes in the concrete, not only in longitudinal, but also in vertical, directions. Horizontal cracks have developed in a number of dams,[21] and such cracks have been observed even at the up-stream face of the Barberine Dam, in Switzerland, the Saint-Marc Dam, in France, and the small Tömmerholt Dam, in Norway. These cracks are obviously caused by non-uniformity in shrinkage and temperature in the mass. Such non-uniformities, therefore, have a significance comparable with that of the tensile stresses found by the author.

Another danger of tensile stresses is introduced by a convex curving of the down-stream face in a vertical plane, as shown in Fig. 28. If principal compressive stresses, n_1, are acting parallel to the face, the convex curving will eventually be followed by tensile stresses, n_2, normal to the face. On the surface of the hatched element, $A\,B\,C\,D$ (Fig. 28), no shearing stresses are acting, and the equilibrium demands that,

$$dn_2\,R\,d\alpha + w\,dz\,R\,d\alpha \sin\alpha = n_1\,dz\,d\alpha$$

or,

$$\frac{dn_2}{dz} = \frac{n_1}{R} - w \sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84)$$

That is, if n_1 is larger than $R\,w \sin\alpha$, $\frac{dn_2}{dz}$ will be positive, and tensile stresses, n_2, will be developed normal to the face, increasing from zero at the very face to a maximum some place in the interior. This maximum value depends on the direction and the magnitude of the first principal stress, n_1, in the interior. The greatest danger of such tensile stresses is obtained for dams with a sharply curved steep down-stream face, that is, for small values of R and $\sin\alpha$.

Such curving of the down-stream face has been suggested in order to reduce the maximum compression in high dams, but a reduction of the compressive stresses accompanied by development of tensile stresses is no better than going from "the frying pan into the fire". Furthermore, the English tests (Fig. 8), indicate that the compressive stresses can be reduced by curving the opposite direction, and this opposite curving shown on Fig. 8 will not introduce any tensile stresses. The suggestion to curve the face, as shown in Fig. 28, is based on the assumption that the classical theory is correct not only for triangular section, but for all sections regardless of shape, which is certainly not true.

[21] "Shrinkage and Cracks in Concrete of Dams," by Fredrik Vogt, Det Kgl. Norske Videnskabers Selskab, Nidaros, Norway, 1930, No. 3.

EUGENE KALMAN,[22] M. AM. SOC. C. E. (by letter).[22a]—In this paper Mr. Jakobsen introduces a new method for studying stress distribution in gravity dams, which like any new method must be judged from two points of view, as expressed in the questions: What is the intrinsic value of the method as applied to problems of structures in general, and of dams in particular; and, what results was the author able to derive in making use of it?

The so-called Ritz method has been applied very successfully in the solution of many problems in modern engineering. Some time ago, this method, which is an adaptation of the Principle of Least Work, was first used in dam theory by the writer[23] as a means of determining radial shear acting between arch slices, or between the cut faces of sections in cantilevers of arch dams.

The writer applied the same method to the analysis of stresses in curved or straight gravity dams at least as early as 1926. Mr. Jakobsen's presentation is the first that has appeared in published form and hence the credit for its introduction into the literature of gravity dams is due him.

As to the results of the paper, there are two distinct fundamental defects in the author's use of the Ritz method, which prevent him from deriving all the value that may be obtained by its proper application.

The principle of least work is applied when the unknown stress distribution is sought in a given structure. Many times, the probable or possible stress distribution can be guessed. Sometimes, two or more different stress distributions appear to be probable or possible. The most correct among them must then be selected in some way. Since a selection is involved, the assumed group of different possible stress distributions may be termed (in accordance with current usage in problems of variation) a "competing" group. The various, assumed stress distributions may be termed, "competing individuals". Making use of this terminology, the principle of least work states that among all individuals of a given competing group, in which the true stress distribution is also included, this true stress distribution yields a smaller value for the work of deformation than any other individual of this group. This formulation of the principle of least work presumes that the true system of stress is included in the competing group, and this condition is generally satisfied in the more elementary applications of the principle. As a simple example consider a beam with two equal spans acted upon by a uniform load. Any numerical value whatever chosen for the reaction at the middle support will be statically consistent with the given system of loads if the corresponding values of the end reactions are selected accordingly.

In other words if the three reactions are $p\,l - \frac{1}{2}X$, X, and $p\,l - \frac{1}{2}X$, respectively (in which, X may vary), the result will always be consistent with the given load. The competing values for the reaction, X, of the middle support include any number whatever between $-\infty$ and $+\infty$. Each of these numbers is an indvidual of the competing group. The corresponding end reactions are defined by the expression, $p\,l - \frac{1}{2}X$. If $X = 0$, for instance, the end reactions would equal $p\,l$, etc.

[22] Visiting Prof., Civ. Eng., California Inst. of Technology, Pasadena, Calif.

[22a] Received by the Secretary, November 1, 1930.

[23] "Sulle piastre curve," R. Rendiconti, Inst. Lomb. Scienze, Serie II, Vol. LXII, Fasc. VI, X.

To obtain the true reaction of the middle support, the principle of least work may be used and thus the reactions are found to be equal $3\,p\,l : 8, 5\,p\,l : 4$, $3\,p\,l : 8$, respectively, a result which may be verified by other means. Hence, the principle of least work permits an absolutely exact selection of reactions because the true one is present in the competitive group of sets.

An entirely different condition arises when the given competitive group of stress systems does not actually contain the true stress system. Then, the principle of least work must be modified so as to state that, of all competing systems, the one that gives the smaller value for the work of deformation than any other stress system in the group under consideration, will constitute the best possible approximation to the real working stress system, unless another group of supplementary stress systems is considered.

In the ordinary problems of elementary statics, in which application of the so-called theorem of Castigliano is involved, there is always a definite number of unknowns present, such as the reactions. Any of the unknowns may vary from $-\infty$ to $+\infty$, as X in the foregoing example. Since the parameter may assume any value whatever it is obvious that this range of possibilities must include the correct one, which is to be determined.

On the other hand, in all problems of advanced statics or elasticity, in which the so-called Ritz method of the principle of least work is applicable, an infinite number of tentatively assumed values in an infinite number of points must be taken into consideration. For instance, in the case cited herein, parameters should be introduced for the values of the unknown stresses at all points of the triangular profile which represents a two-dimensional region containing a number of points designated ∞^2. It is clearly impossible to deal with so many parameters. Hence, Ritz proposed the method of introducing a limited number of parameters or unknowns which will yield an approximation to the true stress system, the exact representation of which would require an infinite number of parameters. The underlying principle is the same as in the case of the approximation of a function by the first n terms of a Fourier or power series.

It is easy to see, therefore, that there is a basic difference between the two. In one case, the designer is sure that the individual selected by application of the method represents the correct solution. In the other case, he is sure that the individual selected does not represent the correct solution; it represents merely an approximation.

This fact raises the question as to how to estimate the degree of approximation and how to secure satisfactory results in all cases involving the application of the Ritz method.

Consider a beam fixed at one end and acted upon by a single load, P, at the free end, as shown in Fig. 29. Assume that the longitudinal stresses are distributed in accordance with the following equation:

$$\delta_x = \left(a \sin \frac{11\,\pi}{h}\,y + b \sin \frac{9\,\pi}{h}\,y\right) x \qquad (85)$$

The parameters, a and b, may be assumed to be any value whatever consistent with the condition of equilibrium for the given load. The most satisfactory combination may be found by applying the principle of least work.

The integral of work will reduce to the simple expression,

$$\int\int \delta_x{}^2\, dx\, dy = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \int_0^c \left(a \sin \frac{11\,\pi}{h} y + b \sin \frac{9\,\pi}{h}\right)^2 x^2\, dx\, dy$$

$$= \int_0^c x^2\, dx \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(a^2 \sin^2 \frac{11\,\pi}{h} y + b^2 \sin^2 \frac{9\,\pi}{h} y + 2\,a\,b \sin \frac{11\,\pi}{h} y \sin \frac{9\,\pi}{h} y\right) dy. \quad (86)$$

The form of Equation (85) indicates that the stresses are in equilibrium. In addition, they must balance the bending moment caused by the load, and this requires that,

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} x \left(a \sin \frac{11\,\pi}{h} y + b \sin \frac{9\,\pi}{h} y\right) y\, dy = P\,x \quad \ldots\ldots\ldots (87)$$

Equation (86) integrated and simplified becomes,

$$\frac{1}{2}\, h\, a^2 + \frac{1}{2}\, h\, b^2 \quad \ldots\ldots\ldots (88)$$

and Equation (87) becomes,

$$\frac{2\,a\,h^2}{11^2\,\pi^2} + \frac{2\,b\,h^2}{9^2\,\pi^2} = P \quad \ldots\ldots\ldots (89)$$

In other words, the most satisfactory combination of a and b is that in which $a + b$ is a minimum, and $9^2\,a^2 - 11^2\,b^2 = \frac{9^2\,11^2\,\pi^2}{2\,h^2}$. Therefore, the proper values are, respectively,

$$a = \frac{9^2}{9^4 + 11^4} \frac{\pi^2}{2\,h^2} P$$

and,

$$b = \frac{11^2}{9^4 + 11^4} \frac{\pi^2}{2\,h^2} P$$

Accordingly, the most logical approximation of the true stress system is expressed by the following equation:

$$\delta_x = \frac{1}{9^4 + 11^4} \frac{\pi^2}{2\,h^2} P \left(9^2 \sin \frac{11\,\pi}{h} y - 11^2 \sin \frac{9\,\pi}{h} y\right) x \quad \ldots\ldots\ldots (90)$$

The curve of this equation as plotted in Fig. 30 is a very complex one. No engineer would ever attempt to apply this kind of distribution as a substitute for Navier's linear law. Nevertheless, of all competing stress systems represented by Equation (85), this is the best approximation to the true stress system.

From this example it follows that even the correct application of the principle of least work may yield the most absurd results. In other words, prudence is required when the principle of least work is used in the form known as the Ritz method. The result thus obtained will be only a more or less satisfactory approximation of the actual conditions. This is in contrast to the result secured by means of the Castigliano theorem.

The principle of least work operates in a manner similar to an automatic machine. It will select the most acceptable item from a group of items put before it; but the quality of the selection will depend on the material offered. It is clear from what has been explained in the foregoing remarks that the results obtained by the author will depend upon the skill used in constructing his competing group of stress systems.

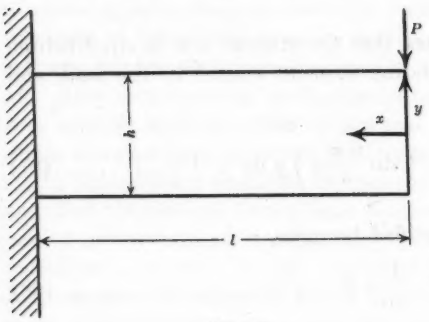

FIG. 29

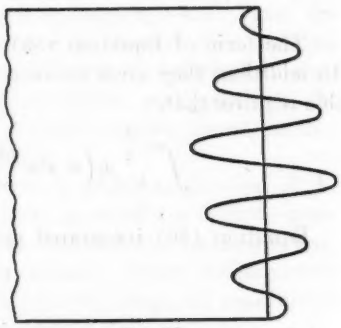

FIG. 30

Mr. Jakobsen's choice of a stress system is represented by Equations (15), (21), and (22). These equations do not define elastic stress systems, because they do not satisfy Equation (43), as given by the author. *Vice versa,* as a matter of fact the stress systems that actually apply in the case of a dam will be those of elasticity. This difference in the characters of the two stress distributions, which may be large quantitatively, constitutes the first fundamental defect, or imperfection in the author's analysis.

A more correct way would have been to assume all competing stresses as elastic systems acting in the profile of the dam. Of course, these should be chosen so as to be statically consistent with the boundary stresses defined in accordance with Equations (15), (21), and (22), by n_y and t along the base of the dam. Each stress system defined by n_y and t on the base, determines one elastic stress system in the profile of the dam and of all these elastic stress systems the most convenient should have been determined by means of the principle of least work.

Of course, there was an extremely difficult problem to be solved, namely, how to determine in all those elastic stress systems the stresses acting in the interior of the dam, thus making possible their substitution in the integral of work and the evaluation of that integral. The author did not attempt to attack this problem, and, consequently, he was forced to assume a certain stress system in the dam, consistent with conditions of stress along the base. That, however, is an arbitrary assumption which might differ radically from actual conditions.

After having selected his competing group of equations, the author computes the work of deformation in the profile of the dam and obtains the stress system which furnishes the smallest value of the work. He then states that this is the most probable stress distribution in the dam. The writer feels,

however, that the result obtained, without considering at all the effect of the foundation, will not represent even a reasonable approximation of the true stress distribution in the dam.

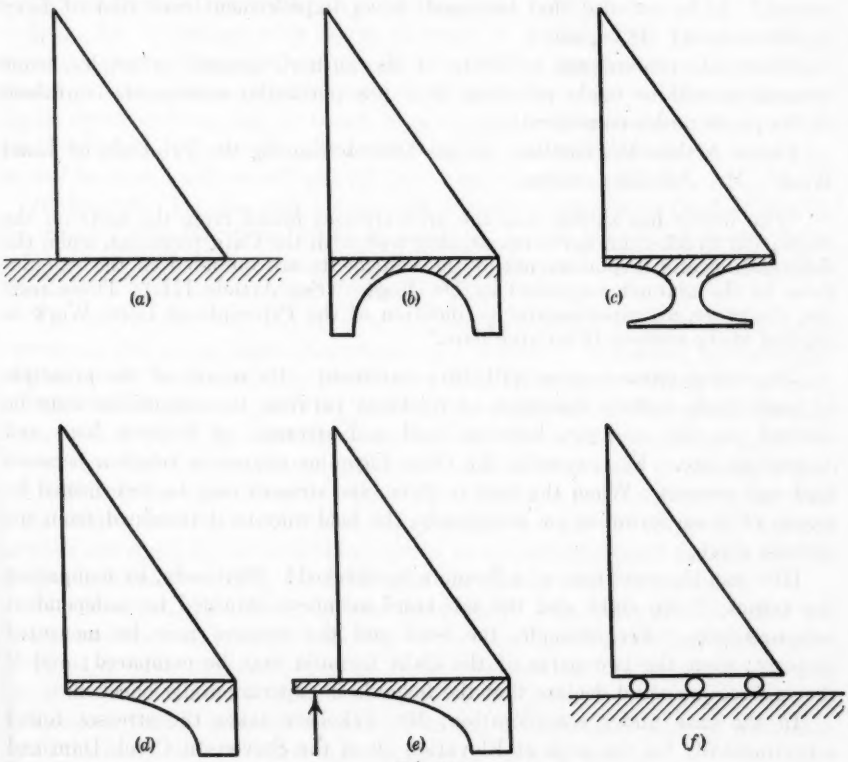

FIG. 31

Evidently, stress conditions in the triangular cross-section of a dam will be entirely different if the dam is considered as a homogeneous solid block with different types of foundation supports as shown, for example, in Fig. 31. Not only will the different structural shape of the supports have its influence on the resulting stress system in the combined or integral section of dam and foundation, but the presence of incidental loads acting directly on the foundation, as shown in Fig. 31, will also have its effect. The author does not consider this phase of the problem. According to him the principle of least work applied to the dam separated from the foundation, furnishes the best stress distribution in the dam, in any case, for any shape of foundation, and for any system of independent load acting directly on the foundation. It is obvious from Fig. 31 that in actual cases the stress distributions will differ completely from each other and from that found by Mr. Jakobsen. This constitutes the second defect in the use of Ritz's method by the author.

The author's aim was to furnish a stress distribution that would be an improvement over that of Lévy, which he claims to be unsatisfactory because

of the deflection conditions in the foundation. He states that Lévy's stress distribution is not acceptable because it implies a sine-shaped curving of the foundation. Will the author's distribution cause the deflection to appear less curved? Is he satisfied that his result is an improvement over that of Lévy in this respect? If so, why?

After this preliminary criticism of the author's general principles, some remarks should be made referring to a few particular statements contained in the paper under consideration.

Under Article IV, entitled "Stress Distribution by the Principle of Least Work", Mr. Jakobsen states,

"The writer has shown that the arch stresses found from the tests on the Stevenson Creek Dam agree remarkably well with the Cain formulas, when the deformation and displacements of the abutments are included in the calculations in the manner suggested by Dr. Vogt. (See Article III.) These tests are, therefore, an experimental verification of the Principle of Least Work as applied to the stresses in an arch dam."

The writer cannot agree with this statement. By means of the principle of least work, certain formulas or relations between two quantities may be derived, as, for example, between load and stresses, or between load and deflections, etc. For example, the Cain formulas express a relation between load and stresses. When the load is given, the stresses may be determined by means of those formulas, or, conversely, the load may be determined from the stresses given.

How can the exactness of a formula be checked? Obviously, by comparing the values of the right and the left-hand members obtained by independent measurements. For example, the load and the stresses may be measured directly; then the two parts of the Cain formula may be compared; and if they agree, one could declare that the formula is experimentally checked.

In the case under consideration, Mr. Jakobsen takes the stresses found experimentally for the arch at Elevation 30 of the Stevenson Creek Dam and compares them with a certain load (depending on several parameters) which he determines by using the Cain formulas. It follows then that the load data at his disposal do not represent measured quantities. They represent values derived by reasoning from the stress data. In the writer's opinion this does not constitute an experimental check of the Cain formula. Mr. Jakobsen has merely applied the Cain formula to evaluate the load.

Suppose, furthermore, that the experiments had revealed variable horizontal load. This might have occurred in the case of an arch dam with central angles of the magnitude of those normally used; that is, central angles considerably larger than those which were unfortunately chosen in the Stevenson Creek Dam.

In that case, the author would follow the same procedure as outlined in his paper and would find no acceptable agreement between the measured stresses and those obtained by Cain's formula which assumes a constant load. Logically, he would then have to state that the principle of least work does not hold in the case of arch dams. On the contrary, however, the principle does hold also in this case as it does in all phenomena thus far encountered in the field

of mechanics. The correct conclusion to be drawn, therefore, would be that the load distribution is different from that implied by the Cain formula.

In Appendix 2, Mr. Jakobsen states that "Equation (27) does not satisfy Equation (43); and, according to Föppl, it should not".

Now, he is dealing with stress systems of a competing group without requiring from them anything except that they be consistent with the given system of load, that is, water pressure and dead load. An infinite number of elastic stress systems may be found that satisfy this prerequisite, all of which would satisfy Equation (43). Consequently, the foregoing statement should be revised to read, "and according to Föppl it need not".

In Appendix III, the author places considerable emphasis on the fact that in the interior of a thin plate which is loaded by lateral forces and is free from any load on the two parallel faces, there may be, or will be, stresses acting normally to the plane of that elementary plate.

The writer believes that it is of no use to attempt to provide for those stresses at the actual beginning stages of a research. Consider, for example, two horizontal sections in the profile of a dam, and assign to them the stresses indicated in Equations (15), (21), and (22), respectively. The two horizontal sections may be a considerable distance apart. It is clear that the nature of the elastic stresses may change infinitely more between these two sections that are relatively far apart than between the two limiting parallel faces of the thin plate mentioned that are infinitely close together. On the latter the normal stresses are equal to zero and this implies an exceedingly small margin for the development of stresses in the interior of the plate normal to the faces.

Nevertheless, the author neglects the possibly wide differences between his system and the true system in the region between those two horizontal sections, while he devotes considerable space to the treatment of the normal stresses in the plate of elementary thickness. The effect is similar to that of making refinements based on possible variation in the fifth decimal place of a number when the numeral in the first decimal place is open to question.

The author states that horizontal sections are not normal sections, and that, therefore, in his opinion, they could not be used as a basis for the application of the trapezoidal rule. He also states that according to his analysis of gravity dams, however, the customary horizontal sections give "practically" the same result as normal sections.

While the concept of a section and the application of Navier's linear distribution law, are entirely satisfactory for beams of limited height, it has no meaning in the analysis of dams. If the stress distribution on the base of a triangular section is assumed to be linear, then the entire elastic system or the condition of stresses in the interior of the section is linear, as stated by Lévy many years ago.[24] Hence, in that case there is no difference at all between normal, horizontal, or inclined sections. The trapezoidal rule applied to any of these sections will furnish the same result exactly, and not only "practically", as the author says.

If conversely the distribution of stresses on the base is assumed not to be linear, then it will not be linear on any section whatever in the interior of the

[24] The writer's exposition in *Beton und Eisen*, 1930, Heft 9, p. 174.

entire profile. The conceptions of the terms "section", "resistance against overturning", etc., are incompatible with modern stress analysis. Mr. Jakobsen deserves credit for having abandoned in the present paper the conception of the section, on which his paper quoted previously was based, introducing instead the stress system defined along every point in the profile by means of Equations (15), (21), and (22).

In Appendix I, the author states:

"The question now arises whether Equation (15) can represent a curve similar to Fig. 10 [which is the stress distribution of Ottley and Brightmore], so that in case this is the correct stress distribution, the constants could be determined so as to give a curve similar to Fig. 10."

Then the author proceeds to determine the constants for that case.

The writer cannot understand the significance of that statement. Since for $y = H$ the author's stress system represents a parabola of the third degree in x, and since every curve whatever may be approximated by such a parabola when the parameters are conveniently selected, it is self-evident that such an approximation may also be made for the Ottley-Brightmore stress curve. Therefore, the writer is unable to understand the reasons for and the aim of such computations in connection with the problem under consideration.

From Equations (25) and (26), the author deduces the fact that the four parameters are independent of the height, H. This fact can be recognized without the use of these equations since it simply follows from the fact that the competing stress system is homogeneous. Using polar co-ordinates instead of Cartesian co-ordinates, all stresses appear to be functions of r and β, in the form of $r\ f\ (\beta)$. In other words, the parameters have nothing to do with r, and will be independent of the longitudinal dimensions of the dam.

The homogeneity of the competing stress system represents a weak point in the author's choice. It has the apparent advantage that it simplifies the computations, but on examination it proves to be one of the less favorable among all possible selections. In fact, since all stresses are of the form $r\ f\ (\beta)$ they correspond to the type of elastic stress distribution defined by Lévy. This is fundamentally in opposition to the conception of the integral profile composed of the dam and the foundation. It involves simply repeating the stresses in the foundation at an ever-increasing rate, due to the presence of an ever-increasing r, while it is evident that the stresses will decrease and disappear toward the interior of the foundation. Stress conditions in the foundations as encountered by the author's system are shown schematically in Fig. 32.

Finally, in his closing remarks Mr. Jakobsen makes the statement that, according to Mach, only actual tests can determine whether or not the principle of least work applies in any specific case.

The writer differs with this opinion in that he believes speculative research is as important as experimental research. They both help to advance science. The author's findings may be compared conveniently with results of tests made by Mr. E. Barnes at the California Institute of Technology in 1929 and 1930. In the writer's opinion, all such tests supply unreliable information as to the actual stress distribution in dams due to the influence of an infinitely

extensive foundation and of the direct water pressure on the rock with its indirect effect on the stresses in the dam itself. These factors are and must be neglected in all these tests.

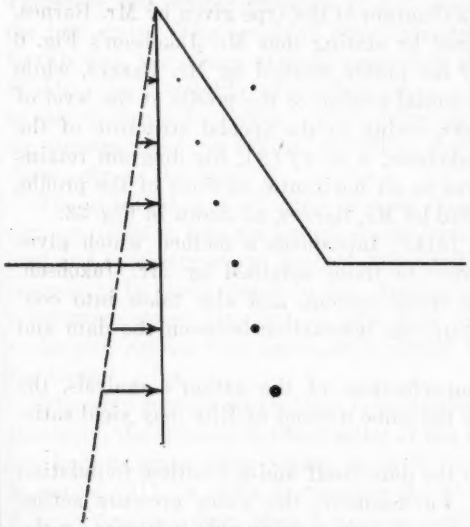

FIG. 32

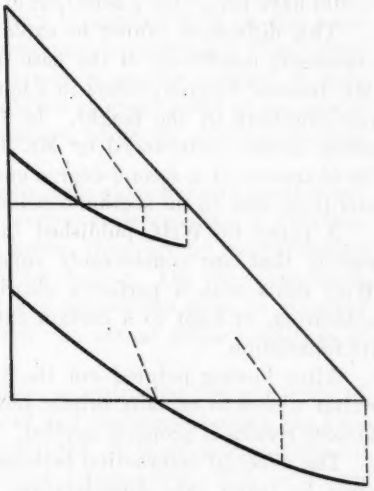

FIG. 33

This opinion is based on rigorous theoretical considerations contained in a previous paper by the writer, entitled "The Validity of Lévy's Elastic Regimen in Gravity Dams",[25] in which paper the deflection curves of the foundation (considered as unlimited), were determined and plotted both with and without the influence of direct water pressure at the bottom of the reservoir. Since the time of its presentation European authorities have shared this opinion fully. This was evident, for example, in a paper on dams presented by Professor Probst at the World Power Conference.[26]

The foregoing general objections refer similarly to the experiments by Mr. Barnes on a model having a moderately wide base, rectangular in shape, and clamped along three sides. The water pressure was represented by a relatively small number of concentrated loads applied normally to the up-stream face.

Mr. Barnes' stress diagram was a kind of regular curve of third degree with a point of contraflexure, whereas Fig. 6, as presented by Mr. Jakobsen, has the general form of a second-degree curve. Both curves agree, inasmuch as they involve an increase of stress at the up-stream face. On the down-stream face, however, Mr. Barnes found an increase of stress, while the author's diagram shows a decrease when compared with the linear distribution. This is a basic difference. Furthermore, the magnitude of the maximum stresses on the up-stream face is quite different in the two diagrams. The author's diagram involves an increase of stress of 50%, while Mr. Barnes' diagram indicates the increase as being equal to 200 per cent.

[25] *Energia Elettrica*, March and April, 1927.

[26] *Zeitschrift* des Vereins Deutscher Ingenieure, 1930, Nos. 13 and 15.

Evidently, one of the two diagrams is entirely in error. The difference in the character of the diagrams is peculiarly striking, because the author disposes of four parameters, that is, of a virtual third-degree parabola, which fact would have permitted a selection of a diagram of the type given by Mr. Barnes.

This difference cannot be explained by stating that Mr. Jakobsen's Fig. 6 represents conditions at the base of the profile studied by Mr. Barnes, while Mr. Barnes' diagram refers to a horizontal section of the profile at the level of one-twentieth of the height. In fact, owing to the special structure of the stress system, introduced by Mr. Jakobsen, $\sigma = r f(\beta)$, his diagram retains its character of a second-degree curve in all horizontal sections of the profile, and thus, also, in the section considered by Mr. Barnes, as shown in Fig. 33.

A paper by Wolf, published in 1914,[27] introduces a method which gives results that are considerably superior to those obtained by Mr. Jakobsen. Wolf deals with a perfectly elastic stress system, and also takes into consideration, at least to a certain extent, the interaction between the dam and its foundation.

After having pointed out the imperfections of the author's analysis, the writer wishes to explain, briefly, how the same method of Ritz may yield satisfactory results if properly applied.

The effect of interaction between the dam itself and a limitless foundation must be taken into consideration. Furthermore, the water pressure acting directly on the bottom of the reservoir will have considerable influence on the deformation of the foundation and, consequently, it will effect the stress distribution in the dam, especially in the more important lower elevations. Hence, the principle of least work must be applied to the entire structure as a unit, including not only the dam, but also the limitless foundation upon which it stands. This phase of the problem must be considered in evaluating the integral of work:

$$A = \frac{1}{2\,G} \int\int \left[\frac{1}{2} (\delta_x{}^2 + \delta_y{}^2) - \frac{1}{2\,(m+1)} (\delta_x + \delta_y)^2 + \tau_{xy}{}^2 \right] dx\, dy \ldots (91)$$

The principle of least work provides for the selection of a stress system from a given group which selection represents the best approximation to the true stresses. In other words, the average of the differences between the true stresses and the selected ones in all points of the dam, is smaller than the average of the differences between the true stresses and those determined by any other stress system of the given group.

It is possible, however, that in certain isolated regions of the profile such differences between the true and the selected stresses may be considerably greater than those between the true stresses and the stresses determined by some other unselected system of the given group. The stress system selected by the principle of least work will supply the best average approximation to the actual stresses; but it may also supply results that represent the actual stresses much less correctly in certain parts of the profile (as, for example, along the base) than some or many others of the systems that were not so selected.

[27] "Zur Integration der Gleichung $\nabla \nabla 0 = 0$ durch Polynome im Falle des Staumauer Problems," *Sitzungsberichte* der Kaiserlichen Akademie der Wissenschaften in Wien, 1914, v. 123, No. 2.

Relatively, designers are not so much interested in conditions at the higher elevations, but rather in the lower parts of the dam, especially along the base. Therefore, the writer proposes to use parameters referring to the points of the base only. The tentative stress system will be applied only on the base. In this way, the source of error previously mentioned is practically eliminated. Besides, the possibility of another group of errors is greatly reduced as may be seen from the following: The author prescribes at every point of the profile the stresses which, in reality, should be determined, while the writer prescribes them only on the base.

The assumption of tentative stresses along the base alone has still another advantage, namely, that the actual elastic stress system corresponding to a certain load and boundary stress is absolutely fixed; hence, there are no indeterminate stresses to be encountered in the interior of the section. On the other hand, a tentative stress distribution with four or five parameters may be expected to furnish results that agree quite closely with the actual stress condition at the base.

The only remaining problem is to determine, according to the principles of elasticity, the stresses in the interior if the load and the stresses along the base are known, since these values must be substituted in their proper place in Equation (91), in order to compute the work of deformation. This is a very difficult problem, even when boundary conditions are most simple. To solve it analytically for the more complicated systems of normal and shearing stresses prescribed along the base would appear to be almost hopeless. Furthermore, the design of gravity dams is not limited to triangular sections. Other shapes, including some that cannot be treated analytically, should be analyzed by some method, and there are no analytical means available for performing such a formidable task.

In his research work the writer reached this stage at the beginning of 1927, and reported on it to J. L. Savage, M. Am. Soc. C. E., of the U. S. Bureau of Reclamation, more than two years ago He determined upon a solution somewhat later, describing it fully to Franklin Thomas, M. Am. Soc. C. E., in July, 1929. The new concept of the exterior work of deformation was substituted for the usual concept of an interior work of deformation. Suppose, for example, that a system of loading is tentatively applied to the dam along the base line. This load will cause stresses in the interior of the section and the work of the deformation will be as expressed in Equation (88). However, the work performed by the interior stresses is equal to that performed by the load. On the other hand, the work performed by the load (that is, by a group of forces), is measured by the sum of products formed by the single forces and the displacement of their points of application in their various directions.

These deflections may be determined experimentally by the use of an elastic model, and thus the work performed by a trial set of forces may be evaluated without the necessity of knowing all the stresses in the interior. Computing the work performed by every individual unit of a competing stress system, both in the dam itself and in the limitless half plane that represents the foundation, the particular stress system that involves the least work will be selected.

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DISCUSSIONS

RELOCATING LAND SURVEYS IN TEXAS

Discussion

BY MESSRS. J. M. HOWE, AND ARTHUR D. KIDDER

J. M. HOWE,[14] M. AM. SOC. C. E.—Texas land-line location is so interwoven with Texas land law that the surveyor or engineer who depends upon the rules of the U. S. General Land Office will soon be completely lost. Some one has said that "land surveying is an art and not a science", and if this applies anywhere, it certainly does in Texas.

The earliest surveys were so placed as to give riparian rights and formed a border along the principal streams with their back lines serrated so as to be roughly parallel to the course of the stream. The later surveys, to fill the unappropriated areas, tie into the riparian grants and make the land map look like an old-fashioned crazy quilt. As the early work was poorly done, with corners and lines inadequately marked, or frequently not marked at all, the fill-in work often laps over or fails to reach the earlier lines. When the first surveys were made the land had values of less than $1 per acre, and scarcely more than that for the later fill-in tracts. In consequence, chaining was carelessly done and courses were only approximate.

The Texas law requires that precedence be given to senior surveys, and many of the conflicts in title are due to failure or inability of the later surveyors to identify the earlier lines. Theoretically, the surveyor should be able to re-locate them by giving the senior surveys their distances and thus retrace the entire map structure on the ground; but original monuments, when identified, hold precedence over both course and distance. Therefore, regardless of the recorded length of lines, when junior surveys are in conflict, the junior must give way. When one appreciates that the early surveyors were working for the grantee and not for the sovereignty of the soil, he can readily understand that many of the early grants have lines on the ground considerably longer than the distances called for in the patent, and it is these lines that hold and not the ones called for in the field notes.

Unfortunately, one has to contend not only with uncertainty of distance, but also with uncertainty of course. North in Texas is seldom true north. The early surveyors, or at least most of them, knew that the magnetic needle had a declination, but their ideas were at considerable variance as to what allowance to make. Not infrequently one will find well identified original lines of surveys in the same county, each calling for the same course, but put

NOTE.—The paper by B. F. Williams, M. Am. Soc. C. E., was presented before the meeting of the Surveying and Mapping Division, Dallas, Tex., April 25, 1929, and published in November, 1930, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[14] Cons. Engr. (Howe & Wise), Houston, Tex.

in by different surveyors and varying by as much as 5° from each other. When the work of different surveyors comes together, the result is what a facetious judge once termed the "Texas rectangle", which he defined as a "rectangular parallelogram, no two sides of which are the same length and no angle of which is a right angle". As far as the speaker knows, no attempt was ever made to allow for earth curvature, and the change in length between meridians due to curvature has to be absorbed in errors in course or distance. This, however, is a minor matter, as the errors in execution of the work are sufficient to make curvature corrections negligible.

In his closing paragraph the author states that the duty of the surveyor is to find and report the facts, and that his duties do not require him to become an advocate. This is true if he is working for the State or qualifying as a witness, but when he is employed by a land owner, his employer expects to be shown the outline of his property and the surveyor is supposed to have sufficient skill to determine these lines. The land owner wants to know the number of acres to which he has title, and what lines he can sustain on the ground.

The surveyor, therefore, must have a working knowledge of the essentials of Texas land law. He must decide in the field, without legal advice, whether he should follow a well marked, and apparently well recognized, crooked line or connect the two corners with a straight line. When calls are inconsistent with measurements on the ground, he must know which calls he should change. Where fencing is found he must know what weight to give to it as fixing the line, and this involves an acquaintanceship with the 3, 5, and 10-year limitation laws and the decisions interpreting them. His success as a surveyor depends upon his ability to place the lines so that they will be sustained in Court.

ARTHUR D. KIDDER,[15] M. AM. SOC. C. E. (by letter).[15a]—The important observations to be noted in this paper are the great need for durable survey monuments, for a complete field-note record of the surveys, and for experience in the work. The need is present both with reference to the establishment of the original boundaries and sub-divisions, and in making retracements of lines for restoration of lost corners and other purposes.

It is made plain that the official survey and plat are the means constituted by law for the identification and description of the public lands of the State of Texas as well as within the Federal domain. The locations are controlled by the established original monuments, and the difficulties of the surveying procedure greatly increase when the monuments disappear. The difficulties become complicated in the event of unexpected discrepancies between the field-note record and the evidence of the survey to be found upon the ground.

The author has not overstressed the need for extended experience in meeting the practical questions which are to be dealt with, the scope of which could only be outlined, and it is presumed to be obvious that a good technical training is requisite, as well as broad field experience. The specialized character of the surveying work described may well be emphasized; similar conditions, with their attendant problems in making restorations, are encountered in other localities.

[15] Assoc. Superv. of Surveys, U. S. Gen. Land Office, Washington, D. C.

[15a] Received by the Secretary, February 10, 1931.